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THREE-DIMENSIONAL FLOWS INSIDE A CYLINDER*

BY

THOMAS C. BENTON

The Pennsylvania State University

Introduction. It is the purpose of this work to derive functions which represent the flow of a perfect fluid inside a cylindrical wall. The velocity perpendicular to this wall must be zero everywhere. Some of the problems discussed will have the axis of the cylinder as an axis of symmetry, but several problems are discussed where the flow is not symmetric about an axis.

In Sec. 1 a certain set of boundary conditions is shown to lead to a series involving Bessel functions. This solution is shown in Sec. 2 to be the result of a point sink at the origin and a circular ring source lying in a plane perpendicular to the axis of the enclosing cylinder and through the origin. This discussion enables the stream function for a source alone and a ring alone to be found. The problem of the combined ring source and point sink in a uniform stream through the cylinder is next discussed. The stream function for a point source in the cylinder not on its axis is then found in Sec. 3.2.

In Sec. 3.4 a new result is obtained which shows that the potential due to a sink not on the axis can be decomposed into a sum of terms, each term being the potential due to a ring source circular in shape but with non-constant strength. It would seem that in many ways these particular ring sources offer the simplest possible configurations with which to begin the study of fields which do not possess axial symmetry.

In Sec. 4 the field of a source and sink on the axis in a stream inside the cylinder is discussed and the semi-axes of the resulting closed stream-surface are computed.

In Sec. 5 the field of the ring with strength $M \cos \theta$ is worked out in detail, this being the least complicated of the ring sources of Sec. 3.4.

1. Let $\psi(\rho, z)$ be a stream function in cylindrical coordinates for a field which is symmetric about the z-axis. Then $\psi(\rho, z)$ is a solution of the equation

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

which has solutions of the type

$$\psi(\rho, z) = \sum_{m=1}^{\infty} A_m \exp(-j_{1m} z/a) \rho J_1(j_{1m}\rho/a),$$

where j_{1m} is the mth positive zero of the Bessel function of the first order $J_1(x)$. It follows

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that $\psi(a, z) \equiv 0$ for all z since $J_1(j_{.m}) = 0$, and $\psi(0, z) \equiv 0$ for all z since ρ is a factor in each term. There is an expansion theorem that if $f(t)$ be a function arbitrarily defined in the interval (0, 1) and

$$\int_0^1 t^{1/2} f(t)\, dt$$

exists, and if

$$a_m = \{2/[J_2(j_{1m})]^2\} \int_0^1 tf(t) J_1(j_{1m}t)\, dt;$$

then

$$\sum_{m=1}^{\infty} a_m J_1(j_{1m}x)$$

converges and its sum is $(1/2)\ \{f(x + 0) + f(x - 0)\}$ at any internal point x of an interval (a, b) such that $0 < a < b < 1$ and $f(t)$ has limited total fluctuation in (a, b).*

Consider the function $F(\rho)$

$$F(\rho) = \begin{cases} M, & 0 < \rho < ka, \quad k < 1 \\ 0, & ka < \rho < a, \end{cases}$$

then $F(\rho)/\rho$ has limited total fluctuation in any interval $0 < \epsilon < b \leq 1$, and

$$\int_0^1 t^{1/2}[F(t)/t]\, dt = \int_0^k Mt^{-1/2}\, dt = 2Mt^{1/2}\Big|_0^k = 2Mk^{1/2}$$

exists (where $t = \rho/a$).

Therefore, the theorem applies to this function and

$$\begin{aligned} a_m &= 2/[aJ_2^2(j_{1m})] \int_0^1 F(at) J_1(j_{1m}t)\, dt, \\ &= 2M/[aJ_2^2(j_{1m})] \int_0^k J_1(j_{1m}t)\, dt, \\ &= 2M[1 - J_0(j_{1m}k)]/aj_{1m}J_2^2(j_{1m}), \end{aligned}$$

since

$$\int J_1(u)\, du = -J_0(u).$$

Now with $A_m = a_m$ the solution for ψ becomes

$$\psi(\rho, z) = p \sum_{m=1}^{\infty} \{2M[1 - J_0(j_{1m}k)]/[aj_{1m}J_2^2(j_{1m})]\} J_1(j_{1m}\rho/a) \exp[(-j_{1m} \mid z \mid)/a] \qquad (1.1)$$

and

$$\begin{aligned} \psi(\rho, 0) &= \rho \sum_{m=1}^{\infty} \frac{2M[1 - J_0(j_{1m}k)]}{aj_{1m}J_2^2(j_{1m})} J_1(j_{1m}\rho/a) \\ &= \rho F(\rho)/\rho = F(\rho) = \begin{cases} M, & 0 < \rho < k\,a \\ 0, & ka < \rho < 1 \end{cases}. \end{aligned}$$

*G. N. Watson, *Bessel functions*, Cambridge Univ. Press, 1922, Sec. 18.24.

It should be noted that $J_2^2(j_{1m}) = J_0^2(j_{1m})$ so that only values of J_0 and J_1 are needed. Hence we have a flow such that $\psi = 0$ consists of the cylinder $\rho = a$ and the part of the plane $z = 0$ between $\rho = ka$ and $\rho = a$, and the z axis. $\psi = M$ consists of the part of $z = 0$ inside $\rho = ka$. The plot of the values of the function shows that the origin is a sink into which the fluid flows and there is a ring source of radius ka in the plane $z = 0$. (Fig. 1* was made for $k = 1/3$, $M = \pi$.)

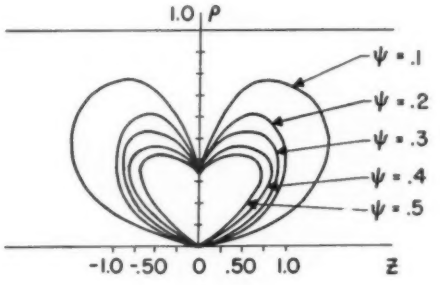

FIG. 1

2. If we take

$$\psi = -\frac{U}{2}\rho^2 + \psi(\rho, z),$$

where $\psi(\rho, z)$ is given by Eq. (1.1), we will have a stream flowing around the ring source and simple sink at its center. If q_z, q_ρ are the velocity components, we have

$$q_z = -\frac{1}{\rho}\frac{\partial \psi}{\partial \rho} = \sum_{m=1}^{\infty} A_m \exp(-j_{1m}z/a) J_0\left(j_{1m}\frac{\rho}{a}\right)\frac{j_{1m}}{a} \tag{2.1}$$

$$q_\rho = \frac{1}{\rho}\frac{\partial \psi}{\partial z} = -\sum_{m=1}^{\infty} A_m \exp(-j_{1m}z/a)\frac{j_{1m}}{a} J_1\left(j_{1m}\frac{\rho}{a}\right). \tag{2.2}$$

If $\rho = 0$, $J_0(0) = 1$, so on the z axis

$$q_z(0, z) = U - \sum_{m=1}^{\infty} \frac{A_m j_{1m}}{a} \exp(-j_{1m}z/a).$$

The stagnation point on the axis will be given by the root of the equation

$$\frac{a^2 U}{2M} = \sum_{m=1}^{\infty} \frac{1 - J_0(j_{1m}k)}{J_0^2(j_{1m})} \exp(-j_{1m}z/a) \tag{2.3}$$

when solved for u and $z = ua$. It will be noted that if a stagnation point can occur at a place not on the z-axis q_z and q_ρ both are zero. So from (2.2) a value can be assigned for z and the resulting equation for ρ can be solved, giving the coordinates of the stagnation point. Then these values can be inserted in (2.1) and the value of U/M can be calcu-

*Calculations were done by the Ordnance Research Laboratory computers. See Appendix I for values of ψ, and Fig. 1.

lated, giving the velocity of the stream at a great distance from the origin. It was found that $z = 0.20$ gives an equation

$$\begin{aligned} f(\rho) \equiv\ & 1.054J_1(3.8817\rho) + 2.620J_1(7.1056\rho) \\ & + 2.825J_1(10.1725\rho) + 1.948J_1(13.3237\rho) \\ & + 0.972J_1(16.4706\rho) + 0.454J_1(19.6159\rho) \\ & + 0.278J_1(22.7601\rho) + 0.226J_1(25.9037\rho) \\ & + 0.167J_1(29.0468\rho) + 0.105J_1(32.1897\rho) \\ & + 0.058J_1(35.3323\rho) + \cdots = 0. \end{aligned}$$

This gives the following table of values:

ρ	0	0.3	0.4	0.49	0.5	0.7
$f(\rho)$	0	2.450	0.668	0.043	−0.020	−0.074

therefore, $z = 0.200$, $\rho = 0.497$ is a stagnation point. This is approximately $z = 0.20$, $\rho = 0.50$. The value of $U/2M = 0.649$ from (2.2); or $U = 1.298m$. Equation (2.3) gives a stagnation point at -0.65 on the z axis. By direct calculation from (2.0) the values of ψ/M can be calculated.*

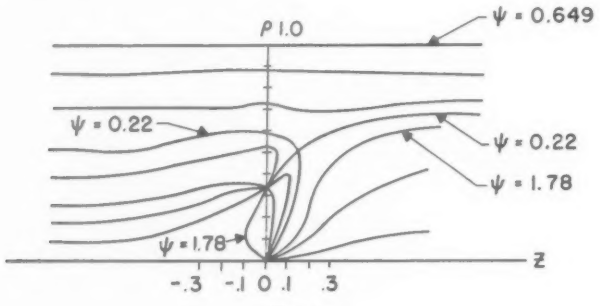

FIG. 2

3. Since the problem of the sink in a cylinder where the flow is along the cylindrical wall is a Neumann problem, it is necessary for $\partial\Phi/\partial n$ to vanish over the cylinder. The velocity potential satisfies the differential equation:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

Let

$$\Phi = e^{-kz}\cos s\varphi P(\rho),$$

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} - \frac{s^2}{\rho^2}P + k^2P = 0,$$

$$\rho^2\frac{d^2P}{d\rho^2} + \rho\frac{dP}{d\rho} + (k^2\rho^2 - s^2)P = 0.$$

Therefore, $P = J_s(k\rho)$.

*See Appendix II for a table of values and Fig. 2.

Let

$$\Phi = \sum_k \sum_{s=0}^{\infty} A_{k,s} e^{-ks} \cos s\varphi J_s(k\rho)$$

$$\partial\Phi/\partial\rho = \sum_k \sum_{s=0}^{\infty} A_{k,s} e^{-ks} (\cos s\varphi) k J_s'(k\rho).$$

Therefore, if $\partial\Phi/\partial\rho = 0$ for $\rho = a$, $J_s'(ka) = 0$. If $h_{s,m}$ is mth root of $J_s' = 0$, $k = h_{s,m}/a$.

3.1 For any axially symmetric field $s = 0$, $J_0'(ka) = -J_1(ka) = 0$; so

$$ka = j_{1,n},$$

$$\Phi = \sum_{n=1}^{\infty} A_n \exp(-j_{1n} \mid z \mid /a) J_0(j_{1n}\rho/a),$$

$$\partial\Phi/\partial z = -\sum_{n=1}^{\infty} A_n (j_{1n}/a) \exp(-j_{1n} \mid z \mid /a) J_0(j_{1n}\rho/a),$$

$$\text{on } z = 0, \quad \frac{\partial\Phi}{\partial z} = \begin{Bmatrix} \infty; & \rho = 0, \text{ simple pole} \\ 0; & 0 < \rho \leq a \end{Bmatrix} = -M\,\delta(\rho),$$

$$\int_0^a \int_0^{2\pi} \left(\frac{\partial\Phi}{\partial z}\right)_0 \rho J_0(j_{1,r}\rho/a)\, d\theta\, d\rho = -2\pi M \quad \text{if} \quad M$$

is the strength of the sink, because

$$\rho\left(\frac{\partial\Phi}{\partial z}\right)_0 = -M\,\delta(\rho).$$

Hence, the integral $= MJ_0(0) \int_0^a\int_0^{2\pi} d\,\theta\, d\rho =$ total flux from the sink on one side of

$$z = 0 = -2\pi A_r \int_0^a (j_{1,r}\rho/a) J_0(j_{1,r}\rho/a) J_0(j_{1,r}\rho/a)\, d\rho,$$

since every integral vanishes except for

$$n = r = -\pi A_{r,j1,r} a J_0^2(j_{1,r})^*,$$

so

$$A_r = 2M/[j_{1,r} a J_0^2(j_{1,r})];$$

hence,

$$\Phi = \sum_{n=1}^{\infty} \{2M/[j_{1n} a J_0^2(j_{1n})]\} \exp(-j_{1n} \mid z \mid /a) J_0(j_{1n}\rho/a)$$

is the velocity potential of some field with a sink at the origin with normal velocity 0 on $\rho = a$.

The stream function is

$$\psi = \sum_{n=1}^{\infty} \{2M/[j_{1n} a J_0^2(j_{1n})]\} \rho J_1(j_{1n}\rho/a) \exp(-j_{1n} \mid z \mid /a), \tag{3.1}$$

*G. N. Watson, *Bessel functions*, Cambridge Univ. Press, 1922, Sec. 5.11 (11).

which obviously gives $\psi(a, z) = 0$ (since $J_1(j_{1n}) = 0$). Also note that this is a part of Eq. (1.1). However, this equation gives a velocity 0 at $z = \infty$, so it is necessary to add a stream flowing from right to left; i.e., a term $M\rho^2/a^2$, then the resulting field is that of a sink in a cylinder.

If a stream of velocity U from right to left is added (Fig. 3):

$$\psi = (|z|/z)(2M\rho/a) \sum_{n=1}^{\infty} \{j_{1n}J_0^2(j_{1n})\}^{-1} J_1(j_{1n}\rho/a) \exp(-j_{1n}|z|/a) + \frac{M\rho^2}{a^2} + \frac{U}{2}\rho^2.$$

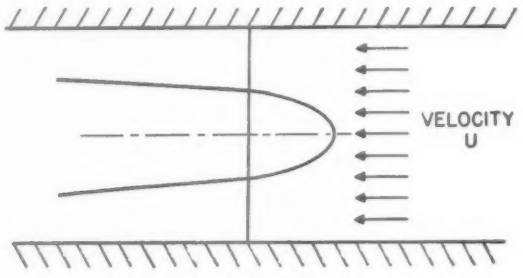

FIG. 3

Assume $a = 1$. Then for $z \geq 0$

$$\psi = 2M\rho \sum_{n=1}^{\infty} \{j_{1n}J_0^2(j_{1n})\}^{-1} J_1(j_{1n}\rho) \exp(-j_{1n}|z|) + \left(M + \frac{U}{2}\right)\rho^2;$$

for $z \leq 0$,

$$\psi = -2M\rho \sum_{n=1}^{\infty} \{j_{1n}J_0^2(j_{1n})\}^{-1} J_1(j_{1n}\rho) \exp(-j_{1n}|z|) + \left(\frac{U}{2} - M\right)\rho^2;$$

$$q_z = -\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} = -\frac{\partial\Phi}{\partial z} = \left\{\sum_{n=1}^{\infty} \{[2MJ_0(j_{1n}\rho)]/J_0^2(j_{1n})\} \exp(-j_{1n}|z|\right\} + 2M - U,$$

$$U > 2M$$

on

$$\rho = 0, \qquad q_z = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} J_0^2(j_{1n}) \exp\left(-j_{1n}|z|\right) = \frac{U}{2M} - 1.$$

Since

$$\frac{J_0^2(j_{1n})}{J_0^2(j_{1,n+1})} \cdot \frac{\exp(-j_{1,n+1}|z|)}{\exp(-j_{1,n}|z|)} \sim \frac{j_{1,n+1}}{j_{1,n}} \exp(-j_{1,n+1} + j_{1,n})|z|$$

$$\cong \left(1 + \frac{\pi}{j_{1n}}\right) \exp(-\pi|z|) \cong \exp(-\pi|z|)$$

as $n \to \infty$, the series converges if $|z| > 0$.

The graph of U/M as a function of z is made from the following Table which was computed from the formula just derived.

z	1	1/2	1/3
U/M	0.150	1.376	3.769

The graph of $U/M = 2 + 2\sum$ can be compared with the function $(U/M)^{1/2}$ which gives the position of the stagnation point for a source in a free stream and it is evident that a stronger stream is needed inside the cylinder to push the stagnation point to the position it has in a free stream. The curves are plotted in Fig. 4.

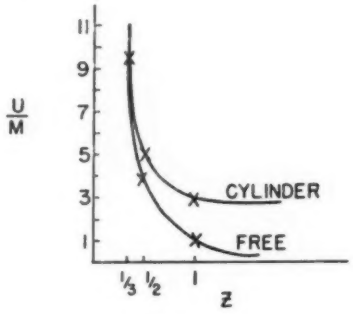

FIG. 4

3.2. In the case where the sink of strength M is at $(b, 0, 0)$ the same method can be used. Let

$$\Phi = \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} A_{ns} \exp(-h_{sn} \mid z \mid /a) J_s(h_{sn}\rho/a) \cos s\rho\varphi, \tag{3.2}$$

where h_{sn} is the nth positive root of $J_s'(z) = 0$. Now

$$(\partial\Phi/\partial z)_{z=0} = -(h_{sn}/a)\Phi(\rho, \varphi, 0).$$

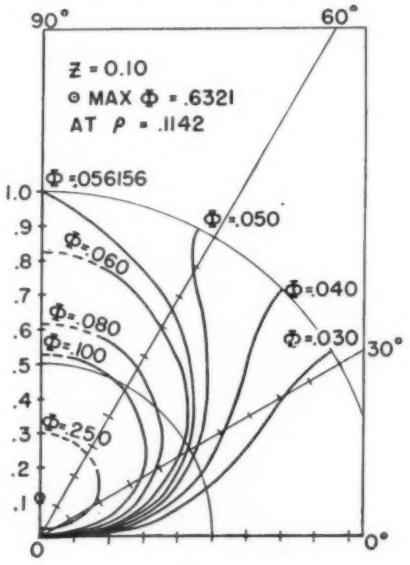

FIG. 5

Hence

$$\int_0^a \int_0^{2\pi} \left(\frac{\partial \Phi}{\partial z}\right)_{z=0} \cos s\varphi J_s(h_{sn}\rho/a)\ \rho\, d\varphi\, d\rho$$
$$= (1 + \delta_{s0})\pi(h_{sn}/a) A_{sn} \cdot (a^2/2)[h_{sn} - \delta^2] h_{sn}^{-2} J_s^2(h_{sn}),$$

where

$$\delta_{s0} = \begin{cases} 1, & s = 0 \\ 0, & s > 0. \end{cases}$$

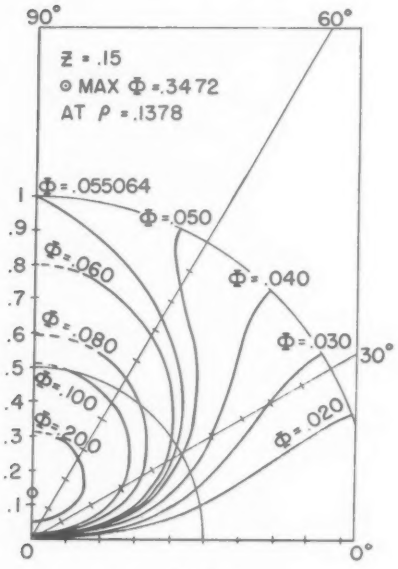

FIG. 6

But $(\partial\Phi/\partial z)_{z=0}$ is zero everywhere except at the point $(b, 0, 0)$, so the integral must be equal to

$$J_s(h_{sn}/a)(-2\pi M).$$

The integral must be interpreted as a Stieltjes Integral of a discontinuous function. Therefore

$$-2\pi M J_s(h_{sn}b/2) = -\pi A_{sn} a(h_{sn}^2 - s^2)(zh_{sn})^{-1} J_s^2(h_{sn})(1 + \delta_{s0}).$$

Therefore

$$A_{sn} = \frac{2Mh_{sn}(2 - \delta_{s0})}{a(h_{sn}^2 - s^2)J_s^2(h_{sn})} J_s(h_{sn}b/a).$$

3.3. In the special case where we demand symmetry with respect to the axis we get a set of terms

$$\Phi = 2Ma^{-1}\left\{\sum_{n=1}^{\infty} j_{1n}^{-1}[J_0(j_{1n})]^{-2} \exp(-j_{1n} \mid z \mid /a) J_0(j_{1n}b/a) J_0(j_{1n}\rho/a)\right\} + 2Mz$$

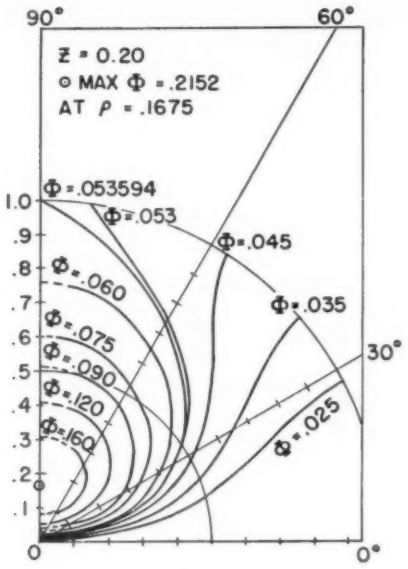

FIG. 7

which still satisfy the differential equation and from (1.1) and (2.1) this must be the potential of a ring sink with $2M$ for its strength and b for its radius, inside a cylinder of radius a after the addition of the stream term. The stream function for this will be

$$\psi = 2Ma^{-1}\left\{\sum_{n=1}^{\infty} j_{1n}^{-1}[J_0(j_{1n})]^{-2} \exp(-j_{1n}z/a)J_0(j_{1n}b/a)J_1(j_{1n}\rho/a)\right\} + Ma^{-2}\rho^2.$$

3.4. A ring source with variable strength $[M/(2\pi b)] \cos s'\varphi_0$ at the point $(b, \varphi_0, 0)$ gives for the potential of the whole ring

$$\Phi_s = \int_0^{2\pi} M(2\pi)^{-1} \cos s'\varphi_0 \left\{\sum_{s=0}^{\infty}\left[\sum_{n=1}^{\infty} A_{sn} \exp(-h_{sn}\,|\,z\,|/a)J_s(h_{sn}\rho/a) \cos s(\varphi - \varphi_0)\right]\right\} d\varphi_0 ,$$

but

$$\int_0^{2\pi} \cos s'\varphi_0 \cos s(\varphi - \varphi_0)\, d\varphi_0 = \begin{cases} 0; & s' \neq s, \\ \pi \cos s\varphi; & s' = s; \end{cases}$$

so

$$\Phi_s = (m/2) \cos s\varphi \left\{\sum_{n=1}^{\infty} A_{sn} \exp(-h_{sn}\,|\,z\,|/a)J_s(h_{sn}\rho/a)\right\}.$$

Since these are exactly the terms of the original series Eq. (3.2.1), which contain $\cos s\varphi$, we have the theorem that the potential Φ for a point source at $(b, 0, 0)$ of strength M on a cylinder consists of the sum of the potential Φ_s for $s = 0, 1, 2, \cdots$ where Φ_s is the potential of a ring source of variable strength in the cylinder and passes through the point source and has a strength $[M/(2\pi b)] \cos s\varphi_0$ at the point $(b, \varphi_0, 0)$ of this ring.

The ring for $s = 1$ will be referred to as a ring of the first order. The ring of zero order is just a ring of constant strength since $\cos s\varphi_0 = 1$ if $s = 0$.

In Sec. 5 below, the field of the first order ring will be worked out in detail.

4. Let there be a source of strength M at $\rho = 0$, $z = b$ and a sink of strength M at $\rho = 0$, $z = -b$ and a stream flowing from right to left inside a cylindrical wall with axis along the z axis. If $z > b$ the streaming motions at infinity due to source and sink cancel each other and these terms may be omitted. The same thing occurs for $z < -b$. But between the two singularities $-b < z < b$ there will be a streaming motion from source to sink.

For $z > b$

$$\psi = (U/2)\rho^2 - \sum_{n=1}^{\infty} 2M\rho j_{1n}^{-1}[J_0(j_{1n})]^{-2}J_1(j_{1n}\rho)\{\exp(-j_{1n} \mid z-b \mid) - \exp(-j_{1n} \mid z+b \mid)\}$$

$$= (U/2)\rho^2 - \sum_{n=1}^{\infty} 2M\rho j_{1n}^{-1}[J_0(j_{1n})]^{-2}J_1(j_{1n}\rho)(\sinh j_{1n}b)\exp(-j_{1n}z).$$

If $\psi = 0$, we will have the dividing stream surface so this surface has the equation

$$\rho^2\left\{U/2 - \sum_{n=1}^{\infty} 2M[J_0(j_{1n})]^{-2}\cdot[2J_1(j_{1n}\rho)/(j_{1n}\rho)](\sinh j_{1n}b)\exp(-j_{1n}z)\right\} = 0.$$

If $\rho \neq 0$, the part of this stream surface, not coincident with the axis $\rho = 0$, has for its equation

$$U/(4M) = \sum_{n=1}^{\infty} [J_0(j_{1n}\rho) + J_2(j_{1n}\rho)][J_0(j_{1n}\rho)]^{-2}(\sinh j_{1n}b)\exp(-j_{1n}z)$$

since $(2/z)\,J_1(z) = J_0(z) + J_2(z)$. Now the points on the axis $\rho = 0$ belonging to this surface are the stagnation points, so

$$U/(4M) = \sum_{n=1}^{\infty} [J_0(j_{1n})]^{-2}(\sinh j_{1n}b)\exp(-j_{1n}z)$$

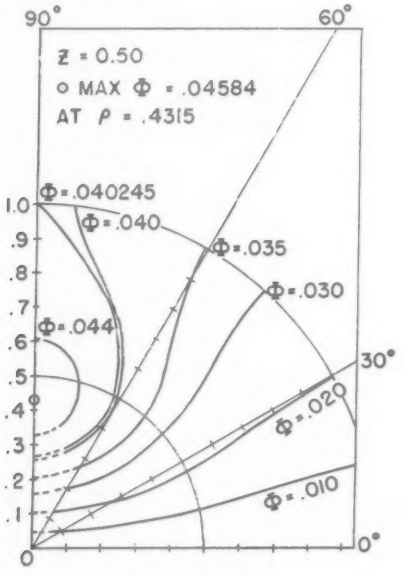

FIG. 8

is an equation for the z coordinate of the stagnation point. This gives the value $U/M = 0.2828$ for $z = 1.50$, and $U/M = 2.6120$ for $z = 1.00$.

For $-b < z < b$,

$$\psi = (U/2)\rho^2 + \sum_{n=1}^{\infty} 2M\rho j_{1n}^{-1}[J_1(j_{1n})]^{-2}J_0(j_{1n}\rho)[\exp(-j_{1n}b)]2\cosh j_{1n}z + 2M\rho^2.$$

Now $\psi = 2M$, $z = 0$ gives the position where the boundary of the closed dividing stream surface crosses the plane $z = 0$. Since $\rho \neq 0$ the equation

$$2M + (U/2) + \sum_{n=1}^{\infty} 2M[J_0(j_{1n})]^{-2}[J_0(j_{1n}\rho) + J_2(j_{1n}\rho)]\exp(-j_{1n}b) = 2M\rho^{-2}$$

gives the intercept. This gives $\rho = 0.955$ for the first case and $\rho = 0.742$ for the second.

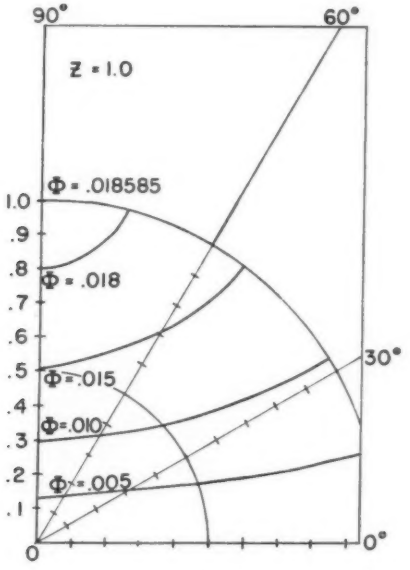

FIG. 9

5. In the case of the most simple ring with variable strength discussed in (3.4), we have

$$\Phi = \tfrac{1}{2}\cos\varphi \sum_{n=1}^{\infty} 2Mh_{1n}J_1(h_{1n}b/a)(h_{1n}^2 - 1)^{-1}[J_1(h_{1n})]^{-2}J_1(j_{1n}\rho/a)\exp(-h_{1n}\,|\,z\,|/a).$$

To obtain some numerical values we will take $b = a/10$, $a = 1$; so that the ring is one tenth of the radius of the cylindrical wall and in a plane perpendicular to the axis of the cylinder with its center on the axis. It is necessary to have the values h_{1n} which are zeros of $J_1'(z)$; since $zJ_1'(z) = zJ_0(z) - J_1(z)$ and the value $z = 0$ is not one in which we

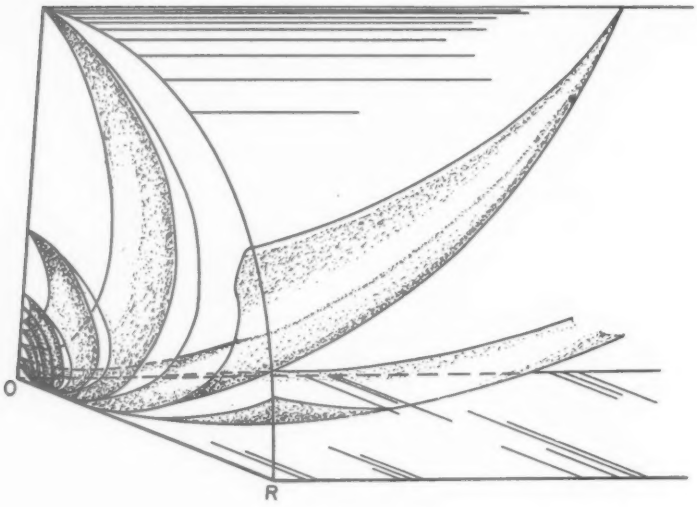

FIG. 10

are interested; these values can be found much more readily from the second form, i.e., $zJ_0(z) = 0$. The values found are as follows

n	1	2	3	4	5	6
h_{1n}	1.8412	5.3314	8.5363	11.7060	14.8636	18.0155
n	7	8	9	10	11	12
h_{1n}	21.1644	24.3113	27.4571	30.6019	33.7462	36.8900

In computing Φ for $z = 0.10$, it was necessary to use 12 terms of the series to get consistent values, a smaller number sufficing for larger values of z.

It will be noted that the strength of the source along the ring varies from a maximum M at $\varphi = 0$, to 0 at $\varphi = \pi/2$, to a minimum $-M$ at $\varphi = \pi$, and back through 0 at $\varphi = 3\pi/2$ to M at $\varphi = 2\pi$. Evidently $\Phi = +\infty$ on the upper half of the ring and $\Phi = -\infty$ on the lower half; and any surface $\Phi = c$ must pass through the points $(1/10, \pi/2, 0)$ and $(1/10, 3\pi/2, 0)$. If c is large and positive, we must obtain a sausage-shaped surface enclosing the upper half of the ring; and if c is large and negative, one enclosing the lower half of the ring. The diametral plane $\varphi = \pi/2, \varphi = 3\pi/2$ is the surface $\Phi = 0$. For small values of Φ there is a closed curve on the cylinder $\rho = a$ which is symmetric to the plane $z = 0$ and to the plane $\Phi = 0$, in which the equipotential surface meets the cylinder orthogonally; and then passes down through the two points of strength 0 on the ring without cutting the plane surface $\Phi = 0$ at any other points. In calculating the values of Φ the values $a = 1.0, 0.5, 0.2, 0.15, 0.1$ were used, and values of ρ from 0 to

1 for about 20 values, so chosen as to give the more interesting situations. Graphs of the equipotentials are given in Figs. 5–9 for each section perpendicular to the axis at the constant values of z in the above series. A sketch of the equipotentials is given in Fig. 10. The streamlines are the curves which are orthogonal to the equipotential surfaces. Since the ring is a curved line-source, the streamlines from any point on the ring must all leave the ring in the plane normal to the ring at that point, and these streamlines form a closed surface which meets the ring again in the point symmetric to the starting point below the surface $\Phi = 0$. These closed surfaces are nested and close down to point limits at the two points of zero strength. They also have as an outer limit, the surface of the bounding cylindrical wall. The streamlines which leave a point on the ring source are shown in Fig. 11.

Acknowledgment. The author wishes to express his thanks to Dr. George F. Wislicenus for directing his attention to the problem and for many helpful discussions as the work progressed. The numerical work would have been almost impossible without the indefatigable aid of the Ordnance Research Laboratory's computing group, particularly Mr. Paul C. Sweetland and Mrs. Barbara R. Kautz. For the figure of the equipotential surfaces of the first order ring, which realizes the form of this solution so completely, the author is deeply indebted to Mrs. Joan P. Lampman.

Appendix I

The stream function

$$\psi(\rho, z) = 2\pi\rho \sum_{m=1}^{\infty} [\exp(-j_m z)][1 - J_0(j_m/3)]j_m^{-1}[J_0(j_m)]^{-2}J_1(j_m\rho)$$

must be evaluated for $0 \leq z$, $0 \leq \rho \leq 1$, where $J_0(z)$ and $J_1(z)$ are the Bessel functions of order zero and one, and j_m is the mth positive zero of $J_1(z)$. Intervals of 0.1 were chosen for ρ and the series was evaluated for $z = .25$ to $z = 2.00$ with intervals of 0.25. This table and that in App. II were computed by the Computing Section of the Ordnance Research Laboratory.

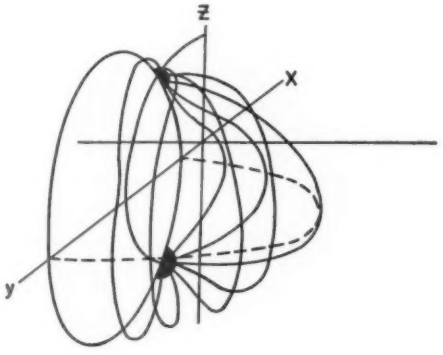

FIG. 11

TABLE 1—ψ

z/ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1	1	1	0	0	0	0	0	0
.25	.1687	.4502	.5639	.4847	.3437	.2262	.1445	.0867	.0415
.50	.0245	.0814	.1360	.1646	.1637	.1426	.1115	.0763	.0393
.75	.0059	.0212	.0396	.0550	.0631	.0628	.0550	.0408	.0221
1.00	.0018	.0067	.0133	.0195	.0239	.0253	.0255	.0181	.0101
1.25	.0006	.0024	.0047	.0072	.0091	.0099	.0094	.0074	.0042
1.50	.0002	.0009	.0018	.0027	.0035	.0038	.0036	.0029	.0016
1.75	.0001	.0003	.0007	.0010	.0013	.0015	.0014	.0012	.0006
2.00		.0001	.0003	.0004	.0005	.0006	.0005	.0004	.0002

Appendix II

Values of ψ/M Computed from Eq. (2.0)

ρ/z	−.30	−.20	−.15	0−	0+	.10	.15	.20	.30
0	0.000	0.000	0.000	−1.000	1.000	0.000	0.000	0.000	0.000
.1		−0.036		−0.994	1.006	0.147		0.049	
.2		−0.067	−0.052	−0.975	1.025	0.187	0.004	0.114	
.3			−0.102	−0.941	1.059		0.219		
.4	0.171	+0.020	0.000	+0.105	0.105	0.237	0.208	0.188	0.091
.5	0.182	0.104	0.082	0.162	0.162	0.211	0.207	0.220	0.182
.6	0.283	0.198	0.204	0.234	0.234	0.259	0.263		0.283
.7		0.260		0.318	0.318		0.264	0.243	
.8				0.415	0.415				
.9				0.525	0.525				
1.0	0.649	0.649	0.649	0.649	0.649	0.649	0.649	0.649	0.649

A few other values are of interest: For $\rho = 0-$, $Z = 1/3-$, $\psi \sim 1.071$, and for $\rho = 0+$, $Z = 1/3+$, $\psi = 0.071$, ψ has a period 2 and lines for which $\psi = a$ or $a + 2$ etc. are the same line, as illustrated by the line through the stagnation point $\psi = 0.22$ or $\psi = -1.78$.

Appendix III

Figures 5–9 are cross sections of the field of the first order ring at distances $z = 0.1$, 0.15, 0.2, 0.5, 1.0 from the plane of the ring, the curves being the equipotentials. These were made from computations done by the ORL Computing Section.

Figure 10 is a sketch of the general form of the equipotentials in one octant of the cylinder. This is the work of Mrs. Joan Lampman.

Figure 11 is a sketch of the stream lines which emanate from one point of the ring source in the case of the first order ring.

ON THE TIME DERIVATIVE OF TENSORS IN MECHANICS OF CONTINUA*

By

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AND

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1. Introduction. In describing the motion of a continuum, not only have spatial coordinates been employed, but sometimes the continuum at time t is referred to a convected or a fixed reference frame. Since tensor fields, such as stress and strain, which arise in the mechanics of continua generally have different transformation laws, the time derivative of these tensors depends not only upon the particular reference frame chosen, but is intimately tied in with the constitutive equations of the medium. Moreover, while the covariant derivative and the time derivative of a tensor commute in a *material* coordinate system, such an interchangeability of order of derivatives is not permissible in *spatial* or even in a *convected* coordinate system.

The question of the time derivative of tensors, especially with reference to the stress tensor, i.e. the *stress rate*, has aroused much interest recently and has been discussed and investigated by Truesdell [1, 2], Noll [3], Oldroyd [4], Thomas [5], Cotter and Rivlin [6], and Prager [7]; the last, with the limitation to rectangular Cartesian coordinates, contains an account of the various definitions for the stress rate proposed in [1, 3, 4, 5, 6], as well as in others, including the earlier work of Jaumann [8][1]. Although the expressions for the stress rate deduced in [1–6] differ from one another, their underlying objective is the requirement that the constitutive equations of the medium involving the stress rate must remain invariant under rigid motions. In addition, Prager [7] has pointed out the desirability for a definition of the stress rate, the vanishing of which will render the invariants of the stress tensor stationary. Also, mention should be made of a very recent paper by Sedov [9], dealing with the time derivative of tensors, which will be referred to again.

The main purpose of the present investigation is to set forth a single general expression for the time derivative (or the time rate) of tensors, valid in all curvilinear coordinates, and applicable to any tensor fields (representing the variables of state in mechanics of continua) whose transformation law is known *a priori*. Indeed, the time derivative of a tensor as derived here is necessarily dependent on the transformation connections (which occur in the transformation law) of the tensor in question, and in particular obeys the same transformation law as the tensor itself. Furthermore, if the rate of a tensor (such as the stress rate), is defined in the context of the present paper, then (for a suitable representation of stress) the following conditions are fulfilled: (a) the stress rate vanishes when the medium executes rigid motion alone and when the stress is referred to a coordinate system participating in this motion, and (b) the rate of invariants of the stress tensor is stationary when the stress rate vanishes. This latter

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[1]As pointed out in [7], Noll's result [3] is identical to that given much earlier by Jaumann [8].

requirement is equivalent to that stipulated by Prager [7] when the coordinate system is rectangular Cartesian and the time derivative is taken in the sense of Jaumann.

Following some preliminary background in Sec. 2, which includes certain results needed in the subsequent analysis, the expression for the time derivative of tensors mentioned above is deduced in Sec. 3 [Eq. (3.13)], where its various parts are conveniently arranged in terms of symbols reminiscent of Christoffel symbols, and is applied to strain as well as stress. The reduction of the latter (i.e., the stress rate) to special cases and its comparison with the previous works, discussed in Sec. 4, are confined, for economy of space, to those in [2, 3, 4, 8, 9]. Finally, in analogy with the time derivative (3.13), an expression for a *coordinate derivative* of tensors is deduced in Sec. 5, which has the properties that (i) when applied to a tensor, it has the same transformation law as the tensor itself, and (ii) it commutes with the time derivative operator (3.13) in all coordinate systems. Aside from its possible future utility, this coordinate derivative of a tensor [Eq. (5.17)] is included here because of the manner in which its development parallels that of (3.13).

2. Preliminaries. Let a generic point of a continuum τ occupy initially (at time $t = t_0$) a position P with the material coordinates X^A, and at time t a position P' with coordinates X'^A; both positions, which may also be identified by position vectors $\mathbf{R}(X)$ and $\mathbf{r}(X', t)$, respectively, are measured relative to the same arbitrary but fixed curvilinear reference frame X. It is clear that while X^A and X'^A refer to the same generic point of the continuum at two different times, they represent two distinct positions in space separated by a displacement $\mathbf{u}$ such that

$$\mathbf{r} = \mathbf{R} + \mathbf{u}. \tag{2.1}$$

If we also introduce an arbitrary curvilinear reference frame x and denote the coordinates of P' in this frame by x^i, then the spatial coordinates x^i and X'^A refer to the same generic point of τ at time t, and the motion of the continuum relative to the fixed frame of reference X may be represented by

$$x^i = x^i(X^B, t); \qquad \left| \frac{\partial x^i}{\partial X^B} \right| > 0, \tag{2.2}$$

or by transformations of the type

$$\begin{aligned} x^i &= x^i(X'^A, t), \\ X'^A &= X'^A(X^B, t), \end{aligned} \tag{2.3}$$

with the restriction that their Jacobians be positive. When only the mapping (2.2) and its inverse are admitted, then the kinematics of continua may be presented as in [10], a description which is also adopted here[2]. Except for some modifications, the notation employed is similar to that in [10]; here, however, covariant differentiation is designated by a stroke (|), while comma is reserved for partial differentiation.

Because of (2.3), tensor transformations from x^i to X'^A have the transformation connections

$$x^i_{,A} \equiv \frac{\partial x^i}{\partial X'^A}, \qquad X'^A_{,i} \equiv \frac{\partial X'^A}{\partial x^i},$$

[2]As will become apparent later, the coordinates X'^A are introduced mainly for added clarity, although it will also offer some advantages.

whereas the tensor transformation laws between x^i and X^A may involve $X^A_{,i}$, r^A_i [see Eqs. (2.7)] or both, as well as their inverses. In what follows, it is convenient to select any one of the three sets (X^A, t), (X'^A, t), and (x^i, t) as independent variables. Thus, if ψ is representative of a tensor of any order, the partial derivatives of ψ with respect to the three space variables are related by expressions of the type

$$\psi_{,i} = \psi_{,A} X^A_{,i} = \psi_{,A'} X'^A_{,i} . \tag{2.4}$$

On the other hand, since the time t is common to all three sets of variables, in taking partial derivative with respect to time, it may be difficult to recognize which of the three sets of variables is taken as independent. To avoid this ambiguity when writing partial derivative with respect to time, it is advisable to specify which coordinates are being held fixed by attaching an appropriate subscript to the partial derivative in question. Accordingly, the partial derivatives of ψ with respect to t will be displayed as

$$\left(\frac{\partial \psi}{\partial t}\right)_X = \psi_{,A'}\left(\frac{\partial X'^A}{\partial t}\right)_X + \left(\frac{\partial \psi}{\partial t}\right)_{X'} \tag{2.5a}$$

$$= \psi_{,i}\left(\frac{\partial x^i}{\partial t}\right)_X + \left(\frac{\partial \psi}{\partial t}\right)_x , \tag{2.5b}$$

and similar expressions for $(\partial\psi/\partial t)_{X'}$ and $(\partial\psi/\partial t)_x$.

For future reference, we recall here that with the use of (2.1) and (2.4) the base vectors of the coordinate curves are related by

$$\begin{aligned} \mathbf{g}_i &= r^A_i \mathbf{G}_A = X'^A_{,i} \mathbf{G}'_A , \\ \mathbf{g}^i &= r^i_A \mathbf{G}^A = x^i_{,A'} \mathbf{G}'^A , \end{aligned} \tag{2.6}$$

where

$$r^i_A = x^i_{,A} - u^i \,|_A , \qquad r^A_i = X^A_{,i} + U^A \,|_i . \tag{2.7}$$

Further, if h denotes an arbitrary vector with components h^i and H^A in the coordinate systems x^i and X^A, respectively, then by (2.6)

$$h^i = r^i_A H^A , \tag{2.8a}$$

while

$$h^i \,|_i = X^B_{,i} r^i_A H^A \,|_B . \tag{2.8b}$$

With the notation

$$\begin{aligned} T^A_B &= \delta^A_B + U^A \,|_B , \\ R^i_i &= \delta^i_i - u^i \,|_i , \end{aligned} \tag{2.9}$$

where δ^i_i is the Kronecker delta, it follows also from (2.7) that

$$\begin{aligned} r^A_i &= T^A_B X^B_{,i} , \\ r^i_A &= R^i_j x^j_{,A} . \end{aligned} \tag{2.10}$$

It is instructive to consider here the velocity of a generic point of the continuum τ. If we select the set (X^A, t) as independent and specify the position vector of this point

at time t by $\mathbf{r} = \mathbf{r}(X^A, t)$, and if we follow the motion of this point (by holding X^A fixed), then the velocity vector is simply

$$\mathbf{v} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_X . \tag{2.11}$$

If, on the other hand, the set (X'^A, t) is taken as independent, in which case $\mathbf{r} = \mathbf{r}(X'^A, t)$, then for an assigned value of X'^A (which may be the final position in space), $\mathbf{r} = \text{const.}$, $(\partial \mathbf{r}/\partial t)_{X'} = 0$, and by (2.5),

$$\mathbf{v} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_X = V'^A \mathbf{G}'_A = \left(\frac{\partial X'^A}{\partial t}\right)_X \mathbf{G}'_A , \tag{2.12}$$

which is an expected result.

While (2.12) represents the *total* (or absolute) velocity of P' relative to the X frame, the total velocity of P' relative to the x frame of reference is not $(\partial x^i/\partial t)_X$, but is given by

$$v^i = \left(\frac{\partial x^i}{\partial t}\right)_X - \left(\frac{\partial x^i}{\partial t}\right)_{X'} \tag{2.13}$$

which is easily verified by application of (2.5a) to x^i. The first term on the right-hand side of (2.13) is the velocity of P' relative to the x frame and will be denoted by

$$q^i = \left(\frac{\partial x^i}{\partial t}\right)_X , \tag{2.14}$$

and in order to assign an interpretation to the second term on the right-hand side of (2.13), we first observe that

$$\left(\frac{\partial x^i}{\partial t}\right)_{X'} = -x^i_{\,,A'}\left(\frac{\partial X'^A}{\partial t}\right)_x , \tag{2.15}$$

where $(\partial X'^A/\partial t)_x$ is easily interpreted as the absolute velocity of a point fixed in the x frame (i.e., a point, the x^i coordinates of which are constant in time). Since $(\partial X'^A/\partial t)_x$ gives the components of this velocity in the X' system, it follows from (2.15) that $(\partial x^i/\partial t)_{X'}$ is the negative of this same velocity in the x frame. Introducing the notation

$$\left(\frac{\partial x^i}{\partial t}\right)_{X'} = -p^i , \tag{2.16}$$

then the combination of (2.13) and (2.14) yields

$$v^i = q^i + p^i , \tag{2.17}$$

a well-known result signifying that the total velocity of P' may be expressed as the sum of its velocity relative to the x frame and the velocity of the x frame itself or more precisely, the velocity of the point in the x frame with which P' is instantaneously in contact.

In the remainder of this section, we dispose of certain results preliminary to our main task. Selecting the set (x^i, t) as independent, then with the aid of (2.16),

$$\left(\frac{\partial \mathbf{g}_{,i}}{\partial t}\right)_x = \mathbf{p}_{,i} = p^k \,|_i \, \mathbf{g}_k , \tag{2.18}$$

and by application of (2.5b) to $X^A_{,i}$, and then to X^A we deduce

$$\left(\frac{\partial X^A_{,i}}{\partial t}\right)_X = -X^A_{,j} q^j_{,i} \,. \tag{2.19}$$

Recalling also that the Jacobian of transformation from x to X is given by [10, p. 63],

$$J = |\, T^A_B \,| = |\delta^A{}_B + U^A|_B| \,, \tag{2.20}$$

then since $(\partial T^A_B/\partial t)_X = (\partial U^A|_B/\partial t)_X$, by (2.1), (2.12), and (2.10) we arrive at

$$\left(\frac{\partial J}{\partial t}\right)_X = \frac{\partial J}{\partial T^A_B}\left(\frac{\partial T^A_B}{\partial t}\right)_X = J(T^{-1})^B_A V^A\,|_B \,. \tag{2.21}$$

Therefore, if in (2.8b) we replace $H^A|_B$ by $V^A|_B$ and introduce the result into (2.21) we obtain the identity

$$\left(\frac{\partial J}{\partial t}\right)_X = J v^i\,|_i \tag{2.22}$$

which will be utilized in Sec. 3.

3. The time derivative of tensors. For the sake of clarity, we first consider the time derivative of a class of tensors with transformation connections of the type r^i_A, and confine attention, without loss in generality, to a mixed second order tensor which transforms according to

$$H^A_B = r^A_i r^j_B h^i_j \,, \tag{3.1}$$

and which may be regarded to include the transformation law appropriate to velocity and displacement components (Sec. 2).

Since there is no difficulty in defining the time derivative of H^A_B in the X frame, i.e. $(\partial H^A_B/\partial t)_X$, the main question confronting us is the definition of the corresponding quantity in the x frame or equivalently the establishment of an appropriate expression for the time derivative (or the time rate) of h^i_j which will be designated by Dh^i_j/Dt. Here this is achieved by seeking the tensor transformation of $(\partial H^A_B/\partial t)_X$ in the sense of the original transformation (3.1), i.e., Dh^i_j/Dt is the desired time derivative provided it transforms according to

$$\frac{Dh^i_j}{Dt} = r^i_A r^B_j \left(\frac{\partial H^A_B}{\partial t}\right)_X \,. \tag{3.2}$$

To this end, we apply $(\partial/\partial t)_X$ to (3.1), multiply throughout by $r^k_A r^B_l$, and by virtue of (3.2) write the resulting expression in the form

$$\frac{Dh^k_l}{Dt} = \left(\frac{\partial h^k_l}{\partial t}\right)_X + r^k_A\left(\frac{\partial r^A_i}{\partial t}\right)_X h^i_l - r^j_B\left(\frac{\partial r^B_l}{\partial t}\right)_X h^k_j \,. \tag{3.3}$$

In (3.3), the coefficient of terms involving h^i_l may be expressed as

$$\begin{aligned} r^k_A\left(\frac{\partial r^A_i}{\partial t}\right)_X &= r^k_A\left[\frac{\partial}{\partial t}(\mathbf{G}^A\cdot\mathbf{g}_i)\right]_X \\ &= \mathbf{g}^k\cdot\left(\frac{\partial \mathbf{g}_i}{\partial t}\right)_X \,, \end{aligned} \tag{3.4}$$

and since by (2.5b), (2.14), and (2.18),

$$\left(\frac{\partial \mathbf{g}_i}{\partial t}\right)_X = \mathbf{g}_{i,j} q^j + p^j \,|_i \, \mathbf{g}_j \,, \tag{3.5}$$

(3.4) becomes

$$r^k_A \left(\frac{\partial r^A_i}{\partial t}\right)_X = \left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} q^j + p^k \,|_i \,. \tag{3.6}$$

Motivated by the fact that the Christoffel symbol of the second kind has the form

$$\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} = \mathbf{g}^i \cdot \mathbf{g}_{i,k} \,,$$

we introduce the notation

$$\binom{i}{j} = \mathbf{g}^i \cdot \left(\frac{\partial \mathbf{g}_j}{\partial t}\right)_X = \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} q^k + p^i \,|_j \,, \tag{3.7}$$

and call $\binom{i}{j}$ the first time symbol. Substitution of (3.7) into (3.3) finally yields the desired result, i.e.,

$$\frac{Dh^k_i}{Dt} = \left(\frac{\partial h^k_i}{\partial t}\right)_X + \binom{k}{i} h^i_l - \binom{j}{l} h^k_j \,, \tag{3.8}$$

which is reminiscent of that for covariant derivative of second order tensors.

A generalization of the foregoing result is necessary, as not all tensors arising in mechanics of continua (especially in the constitutive equations) have transformation laws of the type (3.1). Notable among these are the stress σ^{ij} and the strain e^{ij} which have transformations

$$S^{AB} = J X^A_{,i} X^B_{,j} \sigma^{ij}, \tag{3.9}$$

$$E_{AB} = x^i_{,A} x^j_{,B} e_{ij} \,, \tag{3.10}$$

S^{AB} and E^{AB} being, respectively, the Kirchhoff tensor[3] and a measure of strain in the X frame. In order to distinguish between transformation connections of the type r^i_A and $x^{\ i}_{,A}$, temporarily we use Latin indices for the former and Greek indices for the latter. For our present purposes, it is sufficient to consider a mixed tensor of weight n which transforms according to the law

$$H^{A\Gamma}_{B\Delta} = J^n r^A_i r^i_B X^\Gamma_{,\alpha} x^\beta_{,\Delta} h^{i\alpha}_{j\beta} \,. \tag{3.11}$$

To establish the time derivative of $h^{i\alpha}_{j\beta}$, i.e. $Dh^{i\alpha}_{j\beta}/Dt$, as the tensor transformation of $(\partial H^{A\Gamma}_{B\Delta}/\partial t)_X$ in the sense of (3.11), the procedure is the same as in the case of (3.8), except that in addition to terms involving $\binom{i}{j}$ there also arise expressions $x^\alpha_{,\Lambda}(\partial X^\Lambda_{,\beta}/\partial t)_X$ and $(\partial J/\partial t)_X$ for which use may be made of (2.19) and (2.22), respectively. In addition it is expedient to define the second time symbol by

$$\left[\begin{matrix} \alpha \\ \beta \end{matrix}\right] = x^\alpha_{,\Lambda} \left(\frac{\partial X^\Lambda_{,\beta}}{\partial t}\right)_X = \binom{\alpha}{\beta} - v^\alpha \,|_\beta = -q^{\ \alpha}_{,\beta} \,, \tag{3.12}$$

where (2.17) and (3.7) have been used.

[3]In general, the stress tensor in the initial X frame has different representations. Here S^{AB} is a symmetric tensor.

Thus, omitting the details, the result sought for the time derivative of a tensor with the transformation law (3.11) is given by

$$\frac{Dh^{i\alpha}_{j\beta}}{Dt} = J^{-n} r^{i}_{A} r^{B}_{j} x^{\alpha}_{,\Gamma} X^{\Delta}_{,\beta} \left(\frac{\partial H^{A\Gamma}_{B\Delta}}{\partial t}\right)_X = \left(\frac{\partial h^{i\alpha}_{j\beta}}{\partial t}\right)_X \tag{3.13}$$

$$+ \begin{Bmatrix} i \\ k \end{Bmatrix} h^{k\alpha}_{j\beta} - \begin{Bmatrix} k \\ j \end{Bmatrix} h^{i\alpha}_{k\beta} + \begin{bmatrix} \alpha \\ \mu \end{bmatrix} h^{i\mu}_{j\beta} - \begin{bmatrix} \mu \\ \beta \end{bmatrix} h^{i\alpha}_{j\mu} + n h^{i\alpha}_{j\beta} v^{k}\,|_{k} ,$$

which includes (3.8) as a special case. With reference to the role played by the two time symbols $\begin{Bmatrix} i \\ j \end{Bmatrix}$ and $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ in (3.13), it should be emphasized that the former is associated with the transformation connection r^{i}_{A} and the latter with $x^{\alpha}_{,\Gamma}$. Having distinguished between the two transformation connections by utilizing Latin and Greek indices, in the remainder of the paper we dispense with the latter and use only Latin indices.

Before proceeding further, we consider as an example, the application of (3.13) to the strain tensor

$$e_{ij} = \tfrac{1}{2}[u_i\,|_j + u_j\,|_i - u^m\,|_i\, u_m\,|_j]. \tag{3.14}$$

Since by (3.10) the strain tensor has only transformation connections of the type $x^{i}_{,A}$, then

$$\frac{De_{ij}}{Dt} = \left(\frac{\partial e_{ij}}{\partial t}\right)_X - \begin{bmatrix} k \\ i \end{bmatrix} e_{kj} - \begin{bmatrix} k \\ j \end{bmatrix} e_{ik} . \tag{3.15}$$

Applying (3.13) also to $u_i|_j$, [which by virtue of (2.7) transforms by both r^{i}_{A} and $x^{i}_{,B}$], and after computing $(\partial e_{ij}/\partial t)_X$ and substituting the results into (3.15), simplification yields

$$\frac{De_{ij}}{Dt} = \tfrac{1}{2}(v_i\,|_j + v_j\,|_i) \tag{3.16}$$

for the rate of strain tensor.

Although the application of (3.13) to the stress tensor will be postponed until the next section, it should be emphasized that the time derivative operator defined by (3.13) is applicable to any tensor, whether relative or absolute, provided the transformation law for the tensor in question is known *a priori*. This should not be construed to imply that the application of (3.13) inherently provides for any special requirements which may be demanded of the time derivative of a particular tensor or of the constitutive equations containing it. Indeed, such requirements depend on the transformation connections of the tensor which may even have more than one transformation law; this is exemplified by the representation of stress in the X frame which, in addition to S^{AB}, may also be defined by

$$\sum^{AB} = J r^{A}_{i} X^{B}_{,j} \sigma^{ij}. \tag{3.17}$$

4. The stress rate and comparisons with previous works. As pointed out in [2, 3, 5] and again in [7], an acceptable definition of stress rate must be one such that if the medium executes rigid motion only and if the stress is independent of time when referred to a coordinate system fixed in the medium (i.e., a coordinate system participating in the motion), then the stress rate must vanish when referred to this coordinate system.

It can be easily verified that the time derivative defined by (3.13) automatically satisfies the above criterion when applied to tensors which transform by $x^i_{,A}$ alone, as e.g., in the case of stress with the transformation law (3.9), i.e.

$$\frac{D\sigma^{ij}}{Dt} = \left(\frac{\partial \sigma^{ij}}{\partial t}\right)_x + \begin{bmatrix} i \\ k \end{bmatrix} \sigma^{kj} + \begin{bmatrix} j \\ k \end{bmatrix} \sigma^{ik} + \sigma^{ij} v^k \,|_k \,. \tag{4.1}$$

For, if we let x^i be the coordinate system participating in the rigid motion of the medium, then by (2.14), $q^i = 0$, and (2.17) and (3.12) become

$$v^i = p^i, \qquad \begin{bmatrix} i \\ j \end{bmatrix} = 0, \tag{4.2}$$

respectively. Furthermore, since for a rigid motion $v^k|_k = 0$, the last term in (3.13) vanishes even for relative tensors. Hence, it follows at once from (2.5b) and (4.1) that if ϕ is a representative of a tensor field (such as σ^{ij}) which transforms by $x^i_{,A}$ alone, then

$$\frac{D\phi}{Dt} = \left(\frac{\partial \phi}{\partial t}\right)_x \tag{4.3}$$

which meets the foregoing criterion, stated as condition (a) in Sec. 1.

Moreover, if the rate of stress invariants (such as $\sigma^{ij}\,\sigma_{ij}$) which arise in the theory of plasticity, is defined by application of (3.13), e.g.,

$$\frac{D}{Dt}(\sigma^{ij}\sigma_{ij}) = 2\sigma^{ij}\,\frac{D\sigma_{ij}}{Dt} \tag{4.4}$$

then, clearly, the vanishing of the stress rate will render the left-hand side of (4.4) stationary, and Prager's requirement [condition (b) in Sec. 1] is fulfilled.

If, instead of (3.9), the transformation law (3.17) is adopted for stress, then the application of (3.13) will not yield (4.1). To scrutinize this seemingly paradoxical result, we first observe that by combination of (3.9) and (3.17)

$$\sum^{AB} = T^A_C S^{CB}, \tag{4.5}$$

and that

$$JX^A_{,i} X^B_{,j}\,\frac{D\sigma^{ij}}{Dt} = \left(\frac{\partial S^{AB}}{\partial t}\right)_X \tag{4.6a}$$

$$Jr^A_i x^B_{,j}\,\frac{D\sigma^{ij}}{Dt} = \left(\frac{\partial \sum^{AB}}{\partial t}\right)_X . \tag{4.6b}$$

Next, in conformity with the requirement demanded of a suitable definition of stress rate, we assume at the outset that $(\partial\sigma^{ij}/\partial t)_x = 0$ in the spatial coordinate system (participating in the rigid motion of the medium) for which (4.2) holds, and proceed to determine the partial derivative with respect to t of both $\sum^{AB}$ and S^{AB}. Through consideration of the stress vector and its transformation relation involving the stress tensor and the use of (2.5a), it may be shown that

$$\left(\frac{\partial \sum^{AB}}{\partial t}\right)_X = r^A_i \begin{pmatrix} i \\ k \end{pmatrix} r^k_C \sum^{CB}, \tag{4.7}$$

with the aid of which application of $(\partial/\partial t)_X$ to (4.5) yields

$$\left(\frac{\partial S^{AB}}{\partial t}\right)_X = 0. \tag{4.8}$$

Hence, by the results (4.7) and (4.8), it is clear that *of the two representations for stress in the X frame, only* S^{AB} *with the transformation law* (3.9) *supplies a suitable stress rate*[4]. The time derivative of stress with the transformation law (3.17) specifically leads to

$$\frac{D\sigma^{ij}}{Dt} = \binom{i}{k}\sigma^{kj}$$

which jibes with (4.6b) and (4.7).

In the remainder of this section, we consider the various definitions of stress rate proposed in the recent literature all of which can be derived as special cases of (4.1). In this connection, it is pertinent to mention that since the stress rate (4.1) meets both conditions (a) and (b) discussed above, it follows that any definition of stress rate deduced from (4.1) will also satisfy these conditions.

(a) *Truesdell's definition.* If, as in [2], we select the x frame as one fixed in space, then $p^i = 0$, $q^i = v^i$, and by (3.12)

$$\left[\begin{matrix} i \\ k \end{matrix}\right] = \left\{\begin{matrix} i & \\ k & j \end{matrix}\right\} v^j - v^i\,|_k\,. \tag{4.9}$$

With the aid of (2.5a) and (4.9), (4.1) reduces to

$$\frac{D\sigma^{ij}}{Dt} = \left(\frac{\partial\sigma^{ij}}{\partial t}\right)_x + \sigma^{ij}\,|_k\,v^k - v^i\,|_k\,\sigma^{kj} - v^j\,|_k\,\sigma^{ik} + \sigma^{ij}v^k\,|_k\,. \tag{4.10}$$

Observing that the first two terms on the right-hand side of (4.10) form the material derivative, designated in [2] by a superposed dot, then it becomes immediately apparent that (4.10) is identical with Truesdell's Eq. (1.10) in [2].

(b) *Jaumann's definition.* This definition [8], which has been rediscovered by Noll [3][5], may be deduced as a special case of (4.1) by considering the continuum to undergo rigid motion and referring the rate of stress to a coordinate system fixed in space. Under these conditions by (2.22), $p^i = 0$, and $q^i = v^i$ with

$$v_i\,|_j = \omega_{ij}\,, v^k\,|_k = g^{ki}v_i\,|_k = 0, \tag{4.11}$$

$\omega_{ij} = \frac{1}{2}(v_i|_j - v_j|_i)$ being the vorticity. With the aid of (2.5a), (3.12) and (4.11), (4.1) reduces to

$$\frac{D\sigma^i_j}{Dt} = \left(\frac{\partial\sigma^i_j}{\partial t}\right)_x + \sigma^i_j\,|_k\,v^k - \omega^{ik}\sigma_{kj} + \sigma^{ik}\omega_{kj}\,, \tag{4.12}$$

the result given by Noll [3].

(c) *Oldroyd's definition.* In order to establish the relationship between (3.13) and the corresponding expression in Oldroyd's work [4], it is convenient to consider separately tensor fields which transform by $x^i_{\cdot A}$ and r^i_A. For this purpose, we limit ourselves (without loss in generality) to

$$\sigma^i_j = x^i_{\cdot A}X^{\cdot B}_{\cdot j}S^A_B\,, \tag{4.13}$$

and

$$h^i_j = r^i_A r^B_j H^A_B\,, \tag{4.14}$$

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[4]It is of interest to note that the transformation law for S^{AB} is of the same type as that for strain given by (3.10). This is in keeping with the remark made in Sec. 1 that the choices of stress, strain and their rates are intimately tied in with the constitutive equations of the media.

[5]In this connection, see the remarks made in [7] where other references employing Jaumann's definition are cited.

which have, respectively, the representations

$$S_B^A = X_{,i}^A x_{,B}^i \sigma_i^i \tag{4.15}$$

and

$$H_B^A = X_{,i}^A x_{,B}^i h_i^i , \tag{4.16}$$

in the X coordinate system.

Oldroyd's approach in [4] is to obtain the transformation into the X' system of $[\partial/\partial t\ (\ )]_x$ in the x frame; the latter frame (x) being taken as convected with the medium in which case $q^i = 0$, $p^i = v^i$, and by (3.7) and (3.12)

$$\begin{pmatrix} i \\ j \end{pmatrix} = v^i\,|_i\ , \qquad \begin{bmatrix} i \\ j \end{bmatrix} = 0. \tag{4.17}$$

In order to show the reduction of (3.13) to Oldroyd's results for tensor fields with transformation connections of the type $x_{,A}^i$, we proceed to determine $D\sigma_i^i/Dt$ and DS_B^A/Dt, which (when x^i is convected with the medium), are related through

$$\frac{DS_B'^A}{Dt} = X_{,i}'^A x_{,B'}^i \frac{D\sigma_i^i}{Dt} . \tag{4.18}$$

Taking into account (4.17), (3.13) when applied to σ^{ij} with the aid of (2.5b) yields

$$\frac{D\sigma_i^i}{Dt} = \left(\frac{\partial \sigma_i^i}{\partial t}\right)_X = \left(\frac{\partial \sigma_i^i}{\partial t}\right)_x . \tag{4.19}$$

The definition (3.13) may also be used to calculate $(DS_B'^A/Dt)$ provided we identify x^i in (3.13) with X'^A as utilized in this section. Denoting by V'^A, Q'^A, and P'^A the counterparts[6] of v^i, q^i, and p^i, respectively, in the X' system which is fixed in space, it follows that $P'^A = 0$, $Q'^A = V'^A$, and by (3.7) and (3.12) when x^i is identified with X'^A

$$\begin{aligned} \begin{pmatrix} A \\ B \end{pmatrix}' &= \begin{Bmatrix} A \\ B\ C \end{Bmatrix}' V'^C \\ \begin{bmatrix} A \\ B \end{bmatrix}' &= \begin{Bmatrix} A \\ B\ C \end{Bmatrix}' V'^C - V'^A\,|_{B'} . \end{aligned} \tag{4.20}$$

Applying (3.13) to the combination of (3.7) and (4.15), and utilizing (4.20), (2.5a), and (4.19), we obtain

$$\left(\frac{\partial \sigma_i^i}{\partial t}\right)_x X_{,i}'^A x_{,B'}^i = \left(\frac{\partial S_B'^A}{\partial t}\right)_{X'} + S'^A_B\,|_{C'}\, V'^C + V'^A\,|_{D'}\, S_B'^D + V'^D\,|_{B'}\, S_D'^A . \tag{4.21}$$

The right-hand side of (4.21) is Oldroyd's definition of stress rate which in [4] is denoted by $(\delta S_B'^A/\delta t)$.

We now turn our attention to the reduction of (3.13) to Oldroyd's result when the tensor in question transforms by r_A^i. Following the procedure which led to (4.21), the

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[6] It should be kept in mind that P'^A and Q'^A defined as counterparts of p^i and q^i in the X' system, are not the transformations $(X_{,i}'^A\, p^i)$ and $(X_{,i}'^A\, q^i)$ which hold only if X' and x are two different types of coordinate systems; in Sec. 4 of this paper these two coordinate systems are of the same type.

time derivatives of (4.14) and (4.16) with the aid of (4.17), (4.20), and (2.5a) may be written, respectively, as

$$\frac{Dh_j^i}{Dt} = \left(\frac{\partial h_j^i}{\partial t}\right)_x + v^i\,|_k\, h_j^k - v^k\,|_j\, h_k^i \tag{4.22}$$

and

$$\frac{DH'^A_B}{Dt} = \left(\frac{\partial H'^A_B}{\partial t}\right)_{X'} + H'^A_B\,|_{C'}\, V'^C, \tag{4.23}$$

which may be shown to be related through

$$\frac{Dh_j^i}{Dt} X'^A_{,i} x^j_{,B'} = \frac{DH'^A_B}{Dt}. \tag{4.24}$$

Substitution of (4.22) and (4.23) into (4.24) leads to an equation which when put in the form

$$\left(\frac{\partial h_j^i}{\partial t}\right)_x X'^A_{,i} x^j_{,B'} = \left(\frac{\partial H^A_B}{\partial t}\right)_{X'} + H'^A_B\,|_{C'}\, V'^C - V'^A\,|_{D'}\, H'^D_B + V'^D\,|_{B'}\, H'^A_D \tag{4.25}$$

is similar to (4.21) and agrees with the corresponding result in [4].

In identifying the prediction of (3.13) with Oldroyd's results, no mention was made of the time derivative of relative tensors included in [4]. If the transformation between x and X involves weighting, then a term of the type $v^k|_k$ will occur on both sides of (4.21) and (4.25) and will cancel each other. If the weighting is present in the transformation between x and X', then an additional term, obtained from the determinant $|\,X'^A_{,i}|$, will be present on the right hand side of (4.21) and (4.25) and these will again be in agreement with those in [4] for relative tensors.

(d) *Sedov's results.* The results given in [9] are special cases of (3.13) in two respects: first, as in [4], the coordinate system is convected; and second, in view of the method of approach, the treatment is limited to tensors which transform by r^i_A alone. On account of the latter, only the first time symbols arise which as in (c) above for convected coordinates are given by (4.17) and (4.20).

Since the vanishing of the time derivative (D/Dt) of the base vectors together with (4.17) and (4.20) yield expressions of the type $(\partial \mathbf{g}_i/\partial t)_x = v^k|_i \mathbf{g}_k$, then Sedov's main results expressed in Eqs. (9) of Ref. [9] can be obtained by direct application of (3.13). Thus, for example if h^{ij} transforms by r^i_A alone, we have

$$\frac{Dh^{ij}}{Dt} = \left(\frac{\partial h^{ij}}{\partial t}\right)_x + v^i\,|_k\, h^{kj} + v^j\,|_k\, h^{ik} \tag{4.26a}$$

$$\frac{DH'^{AB}}{Dt} = \left(\frac{\partial H'^{AB}}{\partial t}\right)_x + \left\{\begin{matrix} A \\ C \end{matrix}\ \begin{matrix} \\ D \end{matrix}\right\}' V'^D H'^{CB} + \left\{\begin{matrix} B \\ C \end{matrix}\ \begin{matrix} \\ D \end{matrix}\right\}' V'^D H'^{AC} \tag{4.26b}$$

which are, respectively, the third and fourth Eqs. (9) in Ref. [9]. The first and second of Sedov's Eqs. (9) may be obtained in a similar manner, while in order to deduce the last of Eqs. (9) in [9], we assume as Sedov has that the coordinate system x^i is Cartesian in which case by (3.7),

$$\begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{g}^i \cdot \boldsymbol{\omega} \times \mathbf{g}_j = \omega^i_j\,.$$

The remaining results in [9] are merely specializations of the above and are derived with the further assumption that the coordinate systems x^i and X^A coincide at some particular instant of time.

Before closing this section it should be mentioned that (3.13) also may be shown to reduce to other definitions of stress rate, e.g., that given by Thomas [5], where a special Cartesian coordinate system called "kinematically preferred" is employed. As this reduction is reasonably lengthy, it will not be included here.

5. A coordinate derivative analogous to the time derivative (3.13). While the order of covariant derivatives of a tensor and $[\partial(\ )/\partial t]_X$ is commutative in a material coordinate system, such as interchangeability of order of derivatives, though often useful, is not in general permissible. With the primary aim of providing a mechanism for commutation in the x frame of $D(\ )/Dt$ and an appropriate coordinate derivative, we deduce in this section an expression for the latter which will be designated by double strokes ($||$).

As in Sec. 3, where the time derivative (3.13) is obtained as the tensor transformation of the total time derivative in the X frame, the coordinate derivative that we seek plays a similar role with respect to the covariant derivative, and again depends on the fact that not all tensors transform by the same law from X^A to x^i. For our present purposes, it will suffice to consider separately the following two transformation laws, i.e.,

$$h^i = r^i_A H^A, \tag{5.1}$$

and

$$\sigma^i = x^i_{,A} S^A. \tag{5.2}$$

We first take the partial derivative of (5.1) and with the use of (2.9) write $h^i_{,j}$ as

$$h^i_{,j} = \mathbf{g}^i_{,l} \cdot \mathbf{G}_A H^A + \mathbf{g}^i \cdot \mathbf{G}_{A,B} X^B_{,j} H^A + r^i_A H^A_{,B} X^B_{,j}. \tag{5.3}$$

Recognizing that the left-hand side of this equation and the first term on the right-hand side (with $\mathbf{G}_A H^A = \mathbf{g}_k h^k$), upon transfer, is the covariant derivative of h^i and combining the remaining two terms in the form

$$r^i_A X^B_{,j} \left[H^A_{,B} + \left\{ \begin{matrix} A \\ C \quad B \end{matrix} \right\} H^C \right],$$

we arrive at

$$h^i\,|_j = r^i_A X^B_{,j} H^A\,|_B, \tag{5.4}$$

as the tensor transformation of the covariant derivative of H^A. But, more than this, the transformation law (5.4) also states that the derivative part of $H^A|_B$ transforms by $X^B_{,j}$ and not by r^B_j.

Before proceeding further, we recall [10] the vector sets

$$\begin{aligned} \mathbf{c}_i &= X^A_{,i} \mathbf{G}_A = R^k_i \mathbf{g}_k \\ \mathbf{c}^i &= x^i_{,A} \mathbf{G}^A = (R^{-1})^i_k \mathbf{g}^k \end{aligned} \tag{5.5}$$

and the Cauchy-Green measure of deformation

$$\begin{aligned} c^{ij} &= \mathbf{c}^i \cdot \mathbf{c}^j = x^i_{,A} x^j_{,B} G^{AB}, \\ c_{ij} &= \mathbf{c}_i \cdot \mathbf{c}_j = X^A_{,i} X^B_{,j} G_{AB}, \end{aligned} \tag{5.6}$$

where (2.10) and (2.6) have been employed on the right-hand sides of (5.5). We now turn to (5.2), take its partial derivative, use (5.5) and rearrange the result in a manner similar to (5.4) to obtain

$$\sigma^i_{,j} + \mathbf{c}^i \cdot \mathbf{c}_{k,j}\sigma^k = x^i_{,A}X^B_{,j}S^A\,|_B\,. \tag{5.7}$$

In keeping with the original transformation law (5.2), the left-hand side of (5.7) is the tensor transformation of the covariant derivative $S^A|_B$, where in particular we note that its derivative part transforms by $X^B_{,j}$.

Introducing the notation

$$\left[\left[\begin{matrix} i \\ k \end{matrix}\ \begin{matrix} \\ j \end{matrix}\right]\right] = \mathbf{c}^i \cdot \mathbf{c}_{k,j}\,, \tag{5.8}$$

then the left-hand side of (5.7) may be designated by

$$\sigma^i\,||_j = \sigma^i_{,j} + \left[\left[\begin{matrix} i \\ k \end{matrix}\ \begin{matrix} \\ j \end{matrix}\right]\right]\sigma^k \tag{5.9}$$

which is the appropriate tensor transformation of $S^A|_B$ in the X frame. Since the coordinate derivative (5.9) reduces to the covariant derivative in the X frame, to be consistent, we employ throughout this section the notation $\psi^i||_j$ for the coordinate derivative of any tensor regardless of its transformation law. In view of (5.1) and (5.4), as well as (5.2) and (5.7), it is understood that this coordinate derivative (1) is identical with the covariant derivative $\psi^i|_j$ if Ψ^A transforms by $r^A_{i,}$, and (2) it is defined by the right-hand side of (5.9) with σ^i replaced by ψ^i, if Ψ^A transforms by $X^A_{,i}$.

In order to generalize (5.9) for the coordinate derivative of any tensor, we need to establish the relationship between $\left[\left[\begin{matrix} i \\ j \end{matrix}\ \begin{matrix} \\ k \end{matrix}\right]\right]$ and the Christoffel symbol $\left\{\begin{matrix} i \\ j \end{matrix}\ \begin{matrix} \\ k \end{matrix}\right\}$. To this end, it will suffice to consider the transformation

$$u^i = r^i_A U^A,$$

the first coordinate derivative of which is

$$u^i\,||_j \equiv u^i\,|_j = r^i_A X^B_{,j} U^A\,|_B\,. \tag{5.10}$$

In contrast to the covariant derivative $u^i|_{jk}$, since in (5.10), one index transforms by r^i_A and the other by $X^B_{,j}$, the coordinate derivative $u^i||_{jk}$ reads as

$$\begin{aligned} u^i\,||_{jk} &\equiv u^i\,|_j\,||_k \\ &= u^i\,||_{j,k} + \left\{\begin{matrix} i \\ l \end{matrix}\ \begin{matrix} \\ k \end{matrix}\right\}u^l\,||_j - \left[\left[\begin{matrix} l \\ j \end{matrix}\ \begin{matrix} \\ k \end{matrix}\right]\right]u^i\,||_l \\ &= r^i_A X^B_{,j} X^C_{,k} U^A\,|_{BC}\,, \end{aligned} \tag{5.11}$$

which is obtained by direct computation.

By (5.5) and (5.8), as well as (2.9),

$$\begin{aligned} \left[\left[\begin{matrix} i \\ j \end{matrix}\ \begin{matrix} \\ k \end{matrix}\right]\right] &= (R^{-1})^i_m\left[R^m_l\left\{\begin{matrix} l \\ j \end{matrix}\ \begin{matrix} \\ k \end{matrix}\right\} - u^m\,|_{jk}\right] \\ &= \left\{\begin{matrix} i \\ j \end{matrix}\ \begin{matrix} \\ k \end{matrix}\right\} - (R^{-1})^i_m u^m\,|_{jk}\,. \end{aligned} \tag{5.12}$$

On the other hand, if we solve for $\left[\begin{matrix} i \\ j \quad k \end{matrix}\right]$ from (5.11) with the aid of (2.9), we obtain

$$R^m_i\left[\begin{matrix} i \\ j \quad k \end{matrix}\right] = \left\{\begin{matrix} m \\ j \quad k \end{matrix}\right\} - \left\{\begin{matrix} m \\ l \quad k \end{matrix}\right\} u^l\,|_i - u^m\,||_{ik} + \left\{\begin{matrix} m \\ l \quad k \end{matrix}\right\} u^l\,|_i - \left[\begin{matrix} l \\ j \quad k \end{matrix}\right] u^m\,|_l\,. \tag{5.13}$$

Combination of (5.12) and (5.13), following a lengthy manipulation, yields the relationship between the two symbols $\left[\begin{matrix} i \\ j \quad k \end{matrix}\right]$ and $\left\{\begin{matrix} i \\ j \quad k \end{matrix}\right\}$ in the form

$$\left[\begin{matrix} i \\ j \quad k \end{matrix}\right] = -u^i\,||_{jk} + \left\{\begin{matrix} i \\ j \quad k \end{matrix}\right\} \tag{5.14}$$

which is similar to the difference between the two time symbols given by (3.12).

The relation (5.14) may be used to advantage in obtaining the partial derivative of $J = |\,T^A_B|$. Thus, with the aid of (2.10) and (5.14),

$$J_{,k} = Ju^m\,||_{mk}\,, \tag{5.15}$$

an expression similar to (2.22).

With (5.4) and (5.7) as well as the result (5.15) at our disposal, the generalization of the coordinate derivative for any tensor field is immediate. Again, as in Sec. 3, employing temporarily Latin indices for transformation by r^A_i and Greek indices for transformation by $X^{\ A}_{,i}$, if $H^{A\Gamma}_{B\Delta}$ transforms according to

$$J^n h^{i\alpha}_{j\beta} = r^i_A r^B_{j\ ,\Gamma}\, X^{\ \alpha}_{,\beta} X^{\ \Delta}_{,\beta} H^A_\Delta\,, \tag{5.16}$$

then it can be shown as in the case of (5.4) and (5.7) that the desired coordinate derivative has the form

$$h^{i\alpha}_{j\beta}\,||_k = h^{i\alpha}_{j\beta,k} + \left\{\begin{matrix} i \\ m \quad k \end{matrix}\right\} h^{m\alpha}_{j\beta} - \left\{\begin{matrix} m \\ j \quad k \end{matrix}\right\} h^{i\alpha}_{m\beta} + \left[\begin{matrix} \alpha \\ \mu \quad k \end{matrix}\right] h^{i\mu}_{j\beta} - \left[\begin{matrix} \mu \\ \beta \quad k \end{matrix}\right] h^{i\alpha}_{j\mu} + nh^{i\alpha}_{j\beta} u^m\,||_{mk}\,. \tag{5.17}$$

Dispensing again with the use of Greek indices, and considering the transformations of

$$J^n h^{ik}_{jl}\,||_m \quad \text{and} \quad J^n \frac{Dh^{ik}_{jl}}{Dt}\,,$$

the truth of

$$\left(\frac{Dh^{ik}_{jl}}{Dt}\right)||_m = \frac{D}{Dt}\left(h^{ik}_{jl}\,||_m\right) \tag{5.18}$$

can be verified. That the identity (5.18) holds is not surprising if we recall that (3.13) and (5.17) were deduced, respectively, as the tensor transformations of $(\partial(\ )/\partial t)_x$ and $(\ )|_M$ which commute in the X frame.

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BOOK REVIEWS

Solutions numériques des équations algébriques. By E. Durand. Volume 1. Masson et Cie., Paris, 1960. vii + 327 pp. $13.27.

Whereas the present volume is concerned with the numerical solution of a single equation, a forthcoming second volume is to treat systems of equations. Three introductory chapters (90 pp.) present methods that apply to transcendental as well as algebraic equations (series development, continued fractions, iteration, inverse interpolation, etc.). The principal part of the work, which is concerned with polynominal equations begin with a chapter on computing with polynomials (20 pp.). There follow chapters on transformations of polynomial equations (35 pp.), location of the roots in the complex plane (25 pp.), methods using the powers of the roots (34 pp.), consecutive determination of the roots by iteration (45 pp.), and simultaneous determination of the roots by iteration (14 pp.). A brief chapter on double precision and floating point arithmetic (14 pp.) concludes the volume.

Through out the book the exposition is pragmatical, making only very modest demands on the reader's mathematical background. A wealth of numerical examples illustrate all important features of the methods.

W. Prager

Qualitative theory of differential equations. By V. V. Nemytskii and V. V. Stepanov. Princeton University Press, Princeton, New Jersey, 1960. viii + 523 pp. $12.50.

This volume is a translation of an excellent two-volume Russian work on the qualitative and quantitative theory of differential equations. The first half of the book (Volume One of the original) is devoted to a study of the stability of solutions of linear and nonlinear differential vector equations of the form $dx/dt = A(t)x + f(x, t)$, while the second half covers topological dynamics and ergodic theory.

Both halves of the book are well-written and lucidly presented. The second part will be of most interest since the material of the first part is available in a number of contemporary books in the English language and in translations of the American Mathematical Society.

Richard Bellman

Theory and solution of ordinary differential equations. By Donald Greenspan. The Macmillan Co., New York, 1960. viii + 148 pp. $5.50.

This is an excellent outline for an introductory text in the theory of ordinary differential equations. The only serious weakness as far as the outline is concerned is the lack of matrix techniques in dealing with N-th order linear equations and linear systems.

The author attempts to present in 147 pages, everything from basic concepts of point sets to linear equations to existence theory; from power series solutions to approximate solutions to Sturm-Liouville theory. It can't be done.

Richard Bellman

Mathematical tables. Volume 4: Tables of Weber parabolic cylinder functions and other functions of large argument. By L. Fox. Her Majesty's Stationery Office, London, 1960. iii + 40 pp. $1.85.

The main table gives the Weber functions $W(a, \pm x)$, which occur in the solution of the two-dimensional wave equation in parabolic co-ordinates, in the range $x^{-1} = 0(0.005)\,0.1$, $a = -10(1)10$ to an accuracy of 8 significant figures. In addition, the exponential integral, sine and cosine integrals, Airy integrals, the error integral and the first six factorial functions are tabulated to 10 significant figures for large values of the argument x. Full interpolation facilities are provided.

(*Continued on p. 136*)

ON THE EQUATIONS OF EXTENSIONAL MOTION OF CRYSTAL PLATES*

BY

R. D. MINDLIN AND H. L. COOPER**

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1. Introduction. The Cauchy-Voigt [1] equations of extensional motion of anisotropic, elastic plates may be written as

$$\begin{aligned}(L_1 + \rho\Omega^2)u + L_5 w &= 0,\\ L_5 u + (L_3 + \rho\Omega^2)w &= 0,\end{aligned} \tag{1}$$

where u and w are the components of displacement in the plane of x and z (the plane of the plate), $\Omega = i\partial/\partial t$ and the L_p , $p = 1, 3, 5$, are the quadratic operators,

$$\begin{aligned}L_1 &= \gamma_{11}\frac{\partial^2}{\partial x^2} + 2\gamma_{15}\frac{\partial^2}{\partial x\,\partial z} + \gamma_{55}\frac{\partial^2}{\partial z^2},\\ L_3 &= \gamma_{55}\frac{\partial^2}{\partial x^2} + 2\gamma_{35}\frac{\partial^2}{\partial x\,\partial z} + \gamma_{33}\frac{\partial^2}{\partial z^2},\\ L_5 &= \gamma_{15}\frac{\partial^2}{\partial x^2} + (\gamma_{13} + \gamma_{55})\frac{\partial^2}{\partial x\,\partial z} + \gamma_{35}\frac{\partial^2}{\partial z^2}.\end{aligned} \tag{2}$$

In (2), the γ_{pq} are the elastic stiffnesses relating the stress-resultants (τ_p) and the strains (ϵ_q) according to

$$\tau_p = \gamma_{pq}\epsilon_q\ ; \qquad \gamma_{pq} = \gamma_{qp}\ ; \qquad p, q = 1, 3, 5, \tag{3}$$

where

$$\epsilon_1 = \frac{\partial u}{\partial x}, \qquad \epsilon_3 = \frac{\partial w}{\partial z}, \qquad \epsilon_5 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}.$$

In the case of an isotropic plate,

$$\gamma_{11} = \gamma_{33}\ , \qquad 2\gamma_{55} = \gamma_{11} - \gamma_{13}\ , \qquad \gamma_{15} = \gamma_{35} = 0$$

and Lamé's [2] representation

$$u = \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial z}, \qquad w = \frac{\partial\varphi}{\partial z} + \frac{\partial\psi}{\partial x} \tag{4}$$

reduces solutions of the displacement-equations of motion to solutions of

$$\gamma_{11}\nabla^2\varphi + \rho\Omega^2\varphi = 0, \qquad \gamma_{55}\nabla^2\psi + \rho\Omega^2\psi = 0. \tag{5}$$

In a study of the three-dimensional equations of anisotropic elasticity, G. F. Carrier [3] considered a special elastic symmetry which, in the present context, corresponds to an orthotropic plate ($\gamma_{15} = \gamma_{35} = 0$) with the additional condition

$$(\gamma_{11} - \gamma_{55})(\gamma_{33} - \gamma_{55}) = (\gamma_{13} + \gamma_{55})^2 \tag{6}$$

*Received September 12, 1960.

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and showed that, if

$$u = (\gamma_{11} - \gamma_{55})^{1/2} \frac{\partial \varphi}{\partial x} - (\gamma_{33} - \gamma_{55})^{1/2} \frac{\partial \psi}{\partial z}, \qquad w = (\gamma_{33} - \gamma_{55})^{1/2} \frac{\partial \varphi}{\partial z} + (\gamma_{11} - \gamma_{55})^{1/2} \frac{\partial \psi}{\partial x}, \tag{7}$$

solutions of

$$\gamma_{11} \frac{\partial^2 \varphi}{\partial x^2} + \gamma_{33} \frac{\partial^2 \varphi}{\partial z^2} + \rho \Omega^2 \varphi = 0, \qquad \gamma_{55} \nabla^2 \psi + \rho \Omega^2 \psi = 0 \tag{8}$$

yield, through (7), solutions of the displacement equations of motion.

In this paper, we find the most general representation, of this type, for the Cauchy-Voigt equations; and we show, with the example of quartz, how to find orientations of crystal plates which have constants of elasticity that satisfy the conditions required for reduction of the equations of motion to the simpler forms.

2. Generalization of Carrier's representation in two dimensions. Let

$$u = \left(a_{11} \frac{\partial}{\partial x} + a_{13} \frac{\partial}{\partial z}\right) \chi \equiv M_1 \chi, \qquad w = \left(a_{31} \frac{\partial}{\partial x} + a_{33} \frac{\partial}{\partial z}\right) \chi \equiv M_3 \chi, \tag{9}$$

expecting that there will be two sets of constants a_{ij} and, hence, two functions χ. Then (1) become

$$[(L_1 + \rho\Omega^2) M_1 + L_5 M_3] \chi = 0, \qquad [(L_3 + \rho\Omega^2) M_3 + L_5 M_1] \chi = 0. \tag{10}$$

Because of the presence of the term $\rho\Omega^2$, the only linear operator that can be a factor of the operator in brackets in the first of (10) is M_1 and in the second of (10) is M_3. Hence, if the bracketed operators in (10) are each to be expressed as a product of a linear and a quadratic operator, M_1 must be a factor of $L_5 M_3$ and M_3 must be a factor of $L_5 M_1$. Accordingly, we set

$$M_1 M_3 = L_5 . \tag{11}$$

Upon equating coefficients of like derivatives in (11), we find

$$a_{11} a_{31} = \gamma_{15}, \qquad a_{33} a_{13} = \gamma_{35}, \qquad a_{11} a_{33} + a_{13} a_{31} = \gamma_{13} + \gamma_{55} . \tag{12}$$

If (12) are satisfied, (10) become

$$M_1 (L_1 + M_3^2 + \rho\Omega^2) \chi = 0, \qquad M_3 (L_3 + M_1^2 + \rho\Omega^2) \chi = 0. \tag{13}$$

Now, we require the quadratic operators to be the same in both of (13). Hence, we set

$$L_1 + M_3^2 = L_3 + M_1^2 = N, \quad \text{say}, \tag{14}$$

where

$$N = b_1 \frac{\partial^2}{\partial x^2} + 2 b_5 \frac{\partial^2}{\partial x \, \partial z} + b_3 \frac{\partial^2}{\partial z^2}$$

and the b_p are constants to be determined. Then solutions of

$$(N + \rho\Omega^2)\chi = 0 \tag{15}$$

will yield, through (9), solutions of the Cauchy-Voigt equations.

Upon equating coefficients in (14), we obtain

$$\begin{aligned} \gamma_{11} + a_{31}^2 &= \gamma_{55} + a_{11}^2 = b_1 , \\ \gamma_{33} + a_{13}^2 &= \gamma_{55} + a_{33}^2 = b_3 , \\ \gamma_{15} + a_{33}a_{31} &= \gamma_{35} + a_{11}a_{13} = b_5 . \end{aligned} \tag{16}$$

Equations (12) and (16) contain six relations, among the four a_{ij} and the γ_{pq}, whose consistency requires two relations among the γ_{pq}. Upon elimination of the a_{ij} from (12) and (16), these relations are found to be

$$\begin{aligned} (\gamma_{11} - \gamma_{55})(\gamma_{33} - \gamma_{55}) &= (\gamma_{13} + \gamma_{55})^2 - (\gamma_{15} + \gamma_{35})^2, \\ (\gamma_{13} + \gamma_{55})(\gamma_{15} - \gamma_{35}) &= \gamma_{35}(\gamma_{11} - \gamma_{55}) - \gamma_{15}(\gamma_{33} - \gamma_{55}). \end{aligned} \tag{A}$$

If $\gamma_{15} = \gamma_{35} = 0$, the second of A is satisfied identically and the first of A reduces to (6).

Subject to conditions A, (12) and (16) may be solved for the a_{ij} and b_p. We note, first, that the first two of (12) and the last two of (16) are satisfied by $a_{15} = a_{31}$ if $b_5 \neq 0$ and by $a_{11} = -a_{33}$ if $b_5 = 0$. Hence there are two sets of solutions:

$$\begin{aligned} a_{11} &= \pm\gamma_{15}[(\gamma_{11} - \gamma_{33})/(\gamma_{15}^2 - \gamma_{35}^2)]^{1/2} \\ a_{33} &= \pm\gamma_{35}[(\gamma_{11} - \gamma_{33})/(\gamma_{15}^2 - \gamma_{35}^2)]^{1/2} \\ a_{13} &= a_{31} = \pm[(\gamma_{15}^2 - \gamma_{35}^2)/(\gamma_{11} - \gamma_{33})]^{1/2} \\ b_1 &= \gamma_{11} + (\gamma_{15}^2 - \gamma_{35}^2)/(\gamma_{11} - \gamma_{33}) \\ b_3 &= \gamma_{33} + (\gamma_{15}^2 - \gamma_{35}^2)/(\gamma_{11} - \gamma_{33}) \\ b_5 &= \gamma_{15} + \gamma_{35} \end{aligned} \tag{17}$$

and

$$\begin{aligned} a_{11} &= -a_{33} = \pm i[(\gamma_{15}^2 - \gamma_{35}^2)/(\gamma_{11} - \gamma_{33})]^{1/2} \\ a_{13} &= \pm i\gamma_{35}[(\gamma_{11} - \gamma_{33})/(\gamma_{15}^2 - \gamma_{35}^2)]^{1/2} \\ a_{31} &= \mp i\gamma_{15}[(\gamma_{11} - \gamma_{33})/(\gamma_{15}^2 - \gamma_{35}^2)]^{1/2} \\ b_1 &= b_3 = \gamma_{55} - (\gamma_{15}^2 - \gamma_{35}^2)/(\gamma_{11} - \gamma_{33}) \\ b_5 &= 0 \end{aligned} \tag{18}$$

in which either the upper or lower signs may be taken, without loss of generality.

3. Consequences of conditions A. Conditions A take a simpler form when expressed in terms of the elastic compliances, s_{pq}, defined by

$$\epsilon_p = s_{pq}\tau_q ; \qquad s_{pq} = s_{qp} ; \qquad p, q = 1, 3, 5. \tag{19}$$

and showed that, if

$$u = (\gamma_{11} - \gamma_{55})^{1/2} \frac{\partial \varphi}{\partial x} - (\gamma_{33} - \gamma_{55})^{1/2} \frac{\partial \psi}{\partial z}, \qquad w = (\gamma_{33} - \gamma_{55})^{1/2} \frac{\partial \varphi}{\partial z} + (\gamma_{11} - \gamma_{55})^{1/2} \frac{\partial \psi}{\partial x}, \tag{7}$$

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If $\gamma_{15} = \gamma_{35} = 0$, the second of A is satisfied identically and the first of A reduces to (6).

Subject to conditions A, (12) and (16) may be solved for the a_{ij} and b_p. We note, first, that the first two of (12) and the last two of (16) are satisfied by $a_{13} = a_{31}$ if $b_5 \neq 0$ and by $a_{11} = -a_{33}$ if $b_5 = 0$. Hence there are two sets of solutions:

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From (19) and (3),

$$\gamma_{pq} = S_{pq}/\Delta, \tag{20}$$

where S_{pq} is the cofactor of element s_{pq} in the determinant $\Delta = |s_{pq}|$. When (20) are substituted into conditions A, the latter become

$$s_{11} + s_{33} = 2s_{13} + s_{55}\ , \qquad s_{15} = s_{35} \tag{A'}$$

and we note that conditions A' make

$$\Delta = s_{55}(s_{11}s_{33} - s_{13}^2 - s_{15}^2).$$

Then, with $\varphi = \pm\chi/\Delta^{1/2}$ for (17) and $\psi = \pm i\chi/\Delta^{1/2}$ for (18), we find that (9) and (15) become

$$\begin{aligned} u &= (s_{13} - s_{33})\frac{\partial\varphi}{\partial x} + s_{15}\frac{\partial\varphi}{\partial z} + s_{15}\frac{\partial\psi}{\partial x} + (s_{13} - s_{11})\frac{\partial\psi}{\partial z}\ , \\ w &= s_{15}\frac{\partial\varphi}{\partial x} + (s_{13} - s_{11})\frac{\partial\varphi}{\partial z} - (s_{13} - s_{33})\frac{\partial\psi}{\partial x} - s_{15}\frac{\partial\psi}{\partial z} \end{aligned} \tag{21}$$

and

$$\begin{aligned} s_{33}\frac{\partial^2\varphi}{\partial x^2} - 2s_{15}\frac{\partial^2\varphi}{\partial x\,\partial z} + s_{11}\frac{\partial^2\varphi}{\partial z^2} &= \frac{\rho\,\Delta}{s_{55}}\frac{\partial^2\varphi}{\partial t^2}\ , \\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} &= \rho s_{55}\frac{\partial^2\psi}{\partial t^2}. \end{aligned} \tag{22}$$

We now rotate to new axes x_A , z_A according to

$$x_A = x\cos\Psi_A - z\sin\Psi_A\ , \qquad z_A = x\sin\Psi_A + z\cos\Psi_A \tag{23}$$

in order to eliminate the cross term in the first of (22). The strain-stress relations, referred to axes x_A , z_A are written as

$$\epsilon_p^A = s_{pq}^A\tau_p^A\ ; \qquad p, q = 1, 3, 5,$$

where, with $s = \sin\Psi_A$, $c = \cos\Psi_A$,

$$\begin{aligned} s_{11}^A &= s_{11}c^4 + s_{33}s^4 + (2s_{13} + s_{55})s^2c^2 - 2s_{15}sc^3 - 2s_{35}s^3c, \\ s_{33}^A &= s_{11}s^4 + s_{33}c^4 + (2s_{13} + s_{55})s^2c^2 + 2s_{15}s^3c + 2s_{35}sc^3, \\ s_{55}^A &= 4(s_{11} + s_{33} - 2s_{13})s^2c^2 + s_{55}(c^2 - s^2)^2 + 4(s_{15} - s_{35})(c^2 - s^2)sc, \\ s_{13}^A &= (s_{11} + s_{33} - s_{55})s^2c^2 + s_{13}(s^4 + c^4) + (s_{15} - s_{35})(c^2 - s^2)sc, \\ s_{35}^A &= 2s_{11}s^3c - 2s_{33}sc^3 + (2s_{13} + s_{55})(c^2 - s^2)sc + s_{15}(3s^2c^2 - s^4) + s_{35}(c^4 - 3s^2c^2), \\ s_{15}^A &= 2s_{11}sc^3 - 2s_{33}s^3c - (2s_{13} + s_{55})(c^2 - s^2)sc + s_{15}(c^4 - 3s^2c^2) + s_{35}(3s^2c^2 - s^4). \end{aligned}$$

Now, conditions A' make

$$s_{13}^A = s_{13}\ , \qquad s_{55}^A = s_{55}\ , \qquad s_{11}^A + s_{33}^A = s_{11} + s_{33} \tag{24}$$

and if, in addition, we set

$$\tan 2\Psi_A = 2s_{15}/(s_{33} - s_{11}) \tag{25}$$

we find also

$$s_{11}^A - s_{33}^A = (s_{11} - s_{33}) \sec 2\Psi_A , \qquad s_{15}^A = s_{35}^A = 0. \tag{26}$$

With (21) and (24)–(26), the components of displacement, referred to axes x_A and z_A become

$$u_A = (s_{13}^A - s_{33}^A) \frac{\partial \varphi}{\partial x_A} + (s_{13}^A - s_{11}^A) \frac{\partial \psi}{\partial z_A} ,$$
$$w_A = (s_{13}^A - s_{11}^A) \frac{\partial \varphi}{\partial z_A} - (s_{13}^A - s_{33}^A) \frac{\partial \psi}{\partial x_A}. \tag{27}$$

Also, the differential equations (22) transform to

$$s_{33}^A \frac{\partial^2 \varphi}{\partial x_A^2} + s_{11}^A \frac{\partial^2 \varphi}{\partial z_A^2} = \rho(s_{11}^A s_{33}^A - s_{13}^A s_{13}^A) \frac{\partial^2 \varphi}{\partial t^2} ,$$
$$\frac{\partial^2 \psi}{\partial x_A^2} + \frac{\partial^2 \psi}{\partial z_A^2} = \rho s_{55}^A \frac{\partial^2 \psi}{\partial t^2}. \tag{28}$$

If we were to convert from constants s_{pq}^A to γ_{pq}^A by means of the inverse of (20), we should find that (28) are the same as (8) and that (27) are the same as (7), except for a constant factor. Thus, conditions A reduce the general anisotropic plate to an orthotropic plate satisfying Carrier's condition (6).

4. Slowness vector. Returning to Eqs. (1), we insert the displacements

$$u = A \exp [i\omega(\xi x + \zeta z - t)], \qquad w = C \exp [i\omega(\xi x + \zeta z - t)] \tag{29}$$

corresponding to straight-crested waves. The resulting characteristic equation, which must be satisfied by the components ξ, ζ of the slowness vector [4], is

$$(\gamma_{11}\xi^2 + 2\gamma_{13}\xi\zeta + \gamma_{55}\zeta^2 - \rho)(\gamma_{55}\xi^2 + 2\gamma_{35}\xi\zeta + \gamma_{33}\zeta^2 - \rho)$$
$$- [\gamma_{15}\xi^2 + (\gamma_{13} + \gamma_{55})\xi\zeta + \gamma_{35}\zeta^2]^2 = 0. \tag{30}$$

With ξ, ζ as real Cartesian coordinates, (30) is, in general, the equation of a pair of four-lobed curves. However, if conditions A are satisfied, the discriminant of (30) is a perfect square and the curves are a circle, corresponding to the potential ψ, and an ellipse, corresponding to φ, with principal axes oriented in accordance with (25).

The discriminant of (30) is also a perfect square if

$$(\gamma_{11} - \gamma_{55})(\gamma_{13} + \gamma_{55}) = 2\gamma_{15}(\gamma_{15} - \gamma_{35}),$$
$$(\gamma_{33} - \gamma_{55})(\gamma_{13} + \gamma_{55}) = 2\gamma_{35}(\gamma_{35} - \gamma_{15}). \tag{B}$$

In this case the locus of the tip of the slowness vector is a pair of orthogonal ellipses associated with waves whose properties are described in the following section.

5. Consequences of conditions B. In the case of conditions B, we first perform a rotation to new axes

$$x_B = x \cos \Psi_B - z \sin \Psi_B , \qquad z_B = x \sin \Psi_B + z \cos \Psi_B .$$

Referred to these axes, the stress-strain relation is

$$\tau_p^B = \gamma_{pq}^B \epsilon_q^B ,$$

where, with $s = \sin \Psi_B$, $c = \cos \Psi_B$,

$$\gamma_{11}^B = \gamma_{11}c^4 + \gamma_{33}s^4 + 2(\gamma_{13} + 2\gamma_{55})s^2c^2 - 4\gamma_{15}sc^3 - 4\gamma_{35}s^3c,$$

$$\gamma_{33}^B = \gamma_{11}s^4 + \gamma_{33}c^4 + 2(\gamma_{13} + 2\gamma_{55})s^2c^2 + 4\gamma_{15}s^3c + 4\gamma_{35}sc^3,$$

$$\gamma_{55}^B = (\gamma_{11} + \gamma_{33} - 2\gamma_{13})s^2c^2 + \gamma_{55}(c^2 - s^2)^2 + 2(\gamma_{15} - \gamma_{35})(c^2 - s^2)sc,$$

$$\gamma_{13}^B = (\gamma_{11} + \gamma_{33} - 4\gamma_{55})s^2c^2 + \gamma_{13}(s^4 + c^4) + 2(\gamma_{15} - \gamma_{35})(c^2 - s^2)sc,$$

$$\gamma_{35}^B = \gamma_{11}s^3c - \gamma_{33}sc^3 + (\gamma_{13} + 2\gamma_{55})(c^2 - s^2)sc + \gamma_{15}(3s^2c^2 - s^4) + \gamma_{35}(c^4 - 3s^2c^2),$$

$$\gamma_{15}^B = \gamma_{11}sc^3 - \gamma_{33}s^3c - (\gamma_{13} + 2\gamma_{55})(c^2 - s^2)sc + \gamma_{15}(c^4 - 3s^2c^2) + \gamma_{35}(3s^2c^2 - s^4).$$

Now, if we set $\tan 2\Psi_B = (\gamma_{13} + \gamma_{55})/(\gamma_{35} - \gamma_{15})$ and apply conditions B, we find

$$\gamma_{15}^B = \gamma_{35}^B = \gamma_{13}^B + \gamma_{55}^B = 0.$$

Hence, if u_B and w_B are the components of displacement referred to axes x_B and z_B , the equations of motion become

$$\gamma_{11}^B \frac{\partial^2 u_B}{\partial x_B^2} + \gamma_{55}^B \frac{\partial^2 u_B}{\partial z_B^2} = \rho \frac{\partial^2 u_B}{\partial t^2} , \qquad \gamma_{55}^B \frac{\partial^2 w_B}{\partial x_B^2} + \gamma_{33}^B \frac{\partial^2 w_B}{\partial z_B^2} = \rho \frac{\partial^2 w_B}{\partial t^2} \tag{31}$$

and, consequently, the introduction of displacement-potentials φ and ψ is unnecessary.

Equations (31) have the curious property that, in the waves defined by them, the directions of the respective displacements are fixed: independent of the wave-normals. The mechanism which produces this phenomenon may be identified by noting that, under conditions B,

$$\epsilon_1^B = \gamma_{33}^B[\tau_1^B + (\gamma_{55}^B/\gamma_{33}^B)\tau_3^B]/(\gamma_{11}^B\gamma_{33}^B - \gamma_{55}^B\gamma_{55}^B),$$

$$\epsilon_3^B = \gamma_{11}^B[\tau_3^B + (\gamma_{55}^B/\gamma_{11}^B)\tau_1^B]/(\gamma_{11}^B\gamma_{33}^B - \gamma_{55}^B\gamma_{55}^B).$$

Hence, since the conditions of positive-definiteness of the strain-energy-density require

$$\gamma_{11}^B > 0, \qquad \gamma_{33}^B > 0, \qquad \gamma_{55}^B > 0, \qquad \gamma_{11}^B\gamma_{33}^B - \gamma_{55}^B\gamma_{55}^B > 0,$$

both Poisson's ratios, $\nu_1^B = -\gamma_{55}^B/\gamma_{33}^B$ and $\nu_3^B = -\gamma_{55}^B/\gamma_{11}^B$, must be negative and in the ratios indicated. These are properties unlikely to be found in natural crystals.

6. Application of conditions A to quartz. The constants, say s''_{pq}, of a plate cut from a crystal, at an arbitrary orientation with respect to axes of reference fixed in the crystal, may be expressed in terms of two angles of rotation, say Θ and Φ, and the constants, say s^0_{mn} , $m, n = 1, \cdots 6$, associated with the axes of reference. When the expressions for the s''_{pq} , in terms of the s^0_{mn} , Θ and Φ, are inserted in conditions A', in place of the constants s_{pq} appearing there, a pair of simultaneous, transcendental equations in Θ and Φ is obtained. If there are any real roots of the pair of equations, they give the orientations of plates whose constants s''_{pq} satisfy conditions A'.

In the case of quartz, the compliances associated with reference axes x_0 , y_0 , z_0 , in accordance with the I.R.E. Standards [5], are

$$s^0_{11} = s^0_{22} = 12.77, \qquad s^0_{12} = -1.79,$$
$$s^0_{33} = 9.60, \qquad s^0_{13} = s^0_{23} = -1.22,$$
$$s^0_{44} = s^0_{55} = 20.04, \qquad s^0_{14} = -s^0_{24} = \tfrac{1}{2}s^0_{56} = 4.50,$$
$$s^0_{66} = 2(s^0_{11} - s^0_{12}),$$
$$s^0_{15} = s^0_{16} = s^0_{25} = s^0_{26} = s^0_{34} = s^0_{35} = s^0_{36} = s^0_{45} = s^0_{46} = 0,$$

where the numerical values, in units of 10^{-12} (meter)^2/newton, are those given by Bechmann [6].

A rotation of coordinates about the axis of x_0 to axes

$$x' = x_0 , \qquad y' = y_0 \cos\Theta + z_0 \sin\Theta, \qquad z' = -y_0 \sin\Theta + z_0 \cos\Theta$$

gives formulas for the thirteen non-zero compliances s'_{mn} , $m, n = 1, \cdots 6$, in terms of the s^0_{mn} and Θ, as tabulated by Cady [7].

A further rotation, about the axis of z', to axes

$$x'' = x' \cos\Phi + y' \sin\Phi, \qquad y'' = -x' \sin\Phi + y' \cos\Phi, \qquad z'' = z'$$

gives for the six, of the twenty one, constants which we require:

$$\begin{aligned}
s''_{11} &= s'_{11} \cos^2\Phi + s'_{22} \sin^2\Phi - (s'_{11} + s'_{22} - 2s'_{12} - s'_{66}) \sin^2\Phi \cos^2\Phi,\\
s''_{33} &= s'_{33} ,\\
s''_{55} &= s'_{44} \sin^2\Phi + s'_{55} \cos^2\Phi,\\
s''_{13} &= s'_{13} + (s'_{23} - s'_{13}) \sin^2\Phi,\\
s''_{35} &= s'_{34} \sin\Phi,\\
s''_{15} &= s'_{24} \sin\Phi - (s'_{24} - s'_{14} - s'_{56}) \sin\Phi \cos^2\Phi,
\end{aligned} \tag{32}$$

in which the s'_{mn} are given in terms of the s^0_{mn} and Θ by Cady's [7] formulas.

We now suppose that all of the analysis in Secs. 2 and 3 was performed in the plane of x'' and z'', so that the constants s_{pq} , appearing there, are replaced by s''_{pq} . Then, when (32) are inserted in conditions A', we find

$$[s^0_{14}(3\sin^2\Theta - 4\cos^4\Theta) + 2k\sin^3\Theta\cos\Theta]\sin^3\Phi$$
$$- [s^0_{14}(4\sin^4\Theta - 3\cos^2\Theta) + 2k\sin\Theta\cos^3\Theta]\sin\Phi = 0,$$

$$[2s^0_{14}(3 + \cos^2\Theta)\sin\Theta\cos\Theta - k\sin^4\Theta]\sin^4\Phi$$
$$+ 6[s^0_{14}(2\cos^2\Theta - 1)\sin\Theta\cos\Theta + k\sin^2\Theta\cos^2\Theta]\sin^2\Phi$$
$$- 2s^0_{14}(3 + \sin^2\Theta)\sin\Theta\cos\Theta - k\cos^4\Theta = 0,$$

where $k = s^0_{11} + s^0_{33} - 2s^0_{13} - s^0_{44}$. These equations have the four pairs of roots Θ and Φ given in the first two rows of the accompanying table. The next six rows give the values of the constants s''_{pq} computed from (32). The next row gives the values of ψ_A computed from (25), with s_{pq} replaced by s''_{pq}; and the remaining rows give the values of the constants s^A_{pq} , referred to the axes x_A and z_A defined in (23).

The first column in the table describes the well known "Z-cut" of quartz for which the Cauchy-Voigt equations reduce to isotropic form; i.e., the constants satisfy the conditions

$$s''_{11} = s''_{33}\,, \qquad s''_{55} = 2(s''_{11} - s''_{13}), \qquad s''_{15} = s''_{35} = 0$$

and Eqs. (4) and (5) apply. This cut has been studied in detail by Petrzilka [8], Bechmann [9] and Ekstein [10]. Modes of vibration of the remaining three cuts may be studied by means of Eqs. (27) and (28).

As anticipated, no cut of quartz has constants of elasticity that satisfy conditions B.

TABLE

Orientations (Θ, Φ, Ψ) *and compliances* (s_{pq}, 10^{-12} *meter*^{2}/*newton*) *of the four cuts of quartz which satisfy conditions* A.

	I	II	III	IV
Θ	90°	−9.65°	18.79°	18.79°
Φ	0	0	58.62°	−58.62°
s''_{11}	12.77	12.77	12.52	12.52
s''_{33}	12.77	9.52	9.77	9.77
s''_{55}	29.12	23.27	23.27	23.27
s''_{13}	−1.79	−0.49	−0.49	−0.49
s''_{35}	0	0	−0.87	0.87
s''_{15}	0	0	−0.87	0.87
Ψ_A	0	0	16.17°	73.83°
s^A_{11}	12.77	12.77	12.77	9.52
s^A_{33}	12.77	9.52	9.52	12.77
s^A_{55}	29.12	23.27	23.27	23.27
s^A_{13}	−1.79	−0.49	−0.49	−0.49
s^A_{35}	0	0	0	0
s^A_{15}	0	0	0	0

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STRESS IN THE VICINITY OF A CRACK IN A THICK ELASTIC PLATE*

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1. Introduction. The determination of the distribution of stress in the vicinity of a crack plays a central part in recent theories of fracture [1], [2] and for that reason is of some technical importance. Much of this work is based upon an analysis of the stress near a circular or "penny-shaped" crack first discussed by Sneddon [3]. In that paper Sneddon reduced the problem of finding the stress near a penny-shaped crack in an infinite elastic solid to the determination of the solution of a pair of dual integral equations. Green [4], recognizing the relation of this problem to that of the electrified disc problem of electrostatics (see, for instance, p. 175 of [5]) suggested a different approach leading to the solution of an integral equation. A solution of this crack problem using the ideas of Green and employing oblate spheroidal coordinates has also been given by Payne [6]. Still another approach involving this latter coordinate system is due to Sack [7].

All of these discussions are confined to the case of an infinite solid and only one of them—that due to Sneddon—appears to lend itself to generalization to the case where the crack is situated in a thick plate. Even then the calculations can only be carried out when the crack is lying in the central plane of the plate with its surfaces initially parallel to those of the plate. In this paper we carry out this extension of Sneddon's method for the case in which the thickness of the plate is δ times the diameter of the circular crack.

As in Sneddon's original paper it is assumed that the equations of the classical (infinitesimal) theory of elasticity hold. The boundary conditions appropriate to two types of problem are formulated in Sec. 2. In Sec. 3 the problem is then reduced to the solution of a pair of dual integral equations which are solved in Sec. 4 by a procedure due to Lebedev and Uflyand [8]. Exact analytical expressions are derived for the solution only in the case of large values of δ—the solution is correct to order δ^{-6} in δ. The procedure to be followed in a numerical discussion of the solution is illustrated in Sec. 6 by the calculation of the shape of the deformed crack.

2. The boundary value problems. We shall determine the distribution of stress in the vicinity of a penny-shaped crack in an elastic plate of finite thickness but infinite radius. The crack will be taken to lie in the central plane of the plate with its surfaces parallel to those of the plate. We shall assume that the deformation occurs as the result of the application of prescribed forces or displacements to the plane boundaries of the plate, these forces being distributed symmetrically about an axis which passes through the center of the crack and is perpendicular to the plane of the crack. If we take the radius of the crack to be a, the crack therefore occupies the region $0 \leq r \leq a$, $z = 0$, where (r, θ, z) are cylindrical coordinates referred to the center of the crack as origin

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and having the z-axis along the axis of symmetry of the problem. In this coordinate system we take the components of the displacement vector to be $(u, 0, w)$—the θ-component being identically zero because of the symmetry—and the non-vanishing components of the stress tensor to be σ_r , σ_θ , σ_z , τ_{rz} . The thickness of the plate will be taken to be $2d$.

In the analysis of the boundary value problems we make use of a technique first introduced by Sneddon [3] whereby a crack problem for the thick plate $-d \leq z \leq d$ is converted into a mixed boundary value problem for the elastic layer $0 \leq z \leq d$.

If we assume that the surfaces of the plate are given a uniform displacement ϵ, then we have the boundary conditions

$$\text{on } z = d: \qquad w = \epsilon, \qquad \tau_{rz} = 0, \qquad r \geq 0. \tag{1}$$

$$\begin{aligned} \text{on } z = 0: \qquad & \bar{\tau}_{rz} = 0, \qquad r \geq 0; \\ & \sigma_z = 0, \qquad 0 \leq r \leq a; \\ & w = 0, \qquad r \geq a. \end{aligned} \tag{2}$$

The penultimate equation of this set is valid if there is no pressure applied to the surfaces of the crack, and the last one is a consequence of the prevailing symmetry. If we take the components of the displacement vector to be $(u, 0, w + \epsilon z/d)$ we see that u and w must be such as to lead to a displacement field satisfying the conditions

$$\text{on } z = 0: \qquad w = 0, \qquad r \geq a; \tag{3}$$

$$\tau_{rz} = 0, \qquad r \geq 0; \tag{4}$$

$$\sigma_z = -\frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} \left(\frac{\epsilon}{d}\right) E, \qquad 0 \leq r \leq a. \tag{5}$$

$$\text{on } z = d: \qquad w = 0, \qquad r > 0; \tag{6}$$

$$\tau_{rz} = 0, \qquad r > 0. \tag{7}$$

In Eq. (3), ν denotes Poisson's ratio and E denotes Young's modulus.

On the other hand, if we assume that the free faces of the plate are pulled by a uniform tension F, we have the boundary conditions (2) again but the conditions (1) are replaced by:

$$\text{on } z = d: \qquad \sigma_z = -F, \qquad \tau_{rz} = 0, \qquad r \geq 0. \tag{8}$$

In this case, if we take the components of the displacement vector to be

$$\left\{u, 0, w - \frac{(1 + \nu)(1 - 2\nu)}{1 - \nu} \left(\frac{Fz}{E}\right)\right\},$$

we must choose the functions u, w to lead to a field satisfying the boundary conditions (3), (4), (7) and the conditions

$$\text{on } z = 0: \qquad \sigma_z = -F, \qquad 0 < r < a. \tag{9}$$

$$\text{on } z = d: \qquad \sigma_z = 0, \qquad r > 0. \tag{10}$$

3. The solution of the equilibrium equations. It is well-known ([9], p.452) that the expressions

$$u = -(1 - 2\nu)^{-1} \frac{\partial^2 \phi}{\partial r\, \partial z}, \tag{11}$$

$$w = (1 - 2\nu)^{-1}\left[2(1 - \nu)\nabla^2\phi - \frac{\partial^2 \phi}{\partial z^2}\right], \tag{12}$$

satisfy the equations of elastic equilibrium and the compatibility relations, provided that ϕ satisfies the biharmonic equation

$$\nabla^4 \phi = 0. \tag{13}$$

Corresponding to these components of displacement we have the expressions

$$\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)}\left\{(2 - \nu)\frac{\partial}{\partial z}(\nabla^2\phi) - \frac{\partial^3 \phi}{\partial z^3}\right\}, \tag{14}$$

$$\tau_{rz} = \frac{E}{(1 + \nu)(1 - 2\nu)}\left\{(1 - \nu)\frac{\partial}{\partial r}(\nabla^2\phi) - \frac{\partial^3 \phi}{\partial r\, \partial z^2}\right\}, \tag{15}$$

for two of the stress components. The remaining two are given by similar expressions. Now the function

$$\phi = a^3 \int_0^\infty \xi^{-2}\left\{\left[2\nu A + \frac{(B + A\xi\,\delta)\xi\zeta \sinh \xi\,\delta}{2\nu \sinh \xi\,\delta + \xi\,\delta \cosh \xi\,\delta}\right]\cosh \xi\zeta - (B + A\xi\zeta)\sinh \xi\zeta\right\} J_0(\xi\rho)\, d\xi, \tag{16}$$

where A and B are arbitrary functions of ξ, and

$$\rho = r/a, \qquad \zeta = z/a, \qquad \delta = d/a \tag{17}$$

satisfies the biharmonic equation (13) and it is readily verified that it also leads to an expression for τ_{rz} which satisfies the boundary conditions (4) and (7).

4. The dual integral equations. We shall consider now the first of the two boundary value problems discussed in Sec. 2. The expression (16) satisfies the boundary conditions (4) and (7). It remains to determine the functions $A(\xi)$, $B(\xi)$ so that the conditions (3), (5) and (6) are also satisfied.

If we substitute from Eq. (16) into Eq. (12) and make use of the condition (6), we find, after some manipulation, that, if the function (16) is such that the boundary condition (6) is satisfied then $A(\xi)$, $B(\xi)$ must be related through the equation

$$(\xi\,\delta + 2\nu \sinh \xi\,\delta \cosh \xi\,\delta)A(\xi) = \sinh^2 \xi\,\delta B(\xi).$$

We may therefore write

$$A = -(1 - \nu)a^{-1}\epsilon\psi(\xi), \qquad B = -(1 - \nu)\frac{\xi\,\delta + 2\nu \sinh \xi\,\delta \cosh \xi\,\delta}{\sinh^2 \xi\,\delta}\, a^{-1}\epsilon\psi(\xi)$$

to obtain the expression

$$\phi = -(1 - \nu)\epsilon a^2 \int_0^\infty \xi^{-2}\psi(\xi)[\{2\nu + \xi\zeta \coth \xi\,\delta\}\cosh \xi\zeta - \{\xi\,\delta \operatorname{cosech}^2 \xi\,\delta + 2\nu \coth \xi\,\delta + \xi\zeta\}\sinh \xi\zeta]J_0(\xi\rho)\, d\xi \tag{18}$$

for the stress function $\phi(r, z)$ occurring in Eqs. (11)–(15). Corresponding to this stress function we have the equations

$$u = -\frac{(1-\nu)\epsilon}{1-2\nu}\int_0^\infty \left[(1-2\nu)\frac{\cosh \xi(\delta-\zeta)}{\sinh \xi\,\delta} - \xi\,\delta\frac{\cosh \xi\zeta}{\sinh^2 \xi\,\delta} - \xi\zeta\frac{\sinh \xi(\delta-\zeta)}{\sinh \xi\,\delta}\right]\psi(\xi)J_1(\xi\rho)\,d\xi, \tag{19}$$

$$w = \frac{2(1-\nu)^2\epsilon}{1-2\nu}\int_0^\infty \left[\sinh \xi(\delta-\zeta) + \xi\,\delta\frac{\sinh \xi\zeta}{\sinh \xi\,\delta} - \xi\zeta\cosh \xi(\delta-\zeta)\right]\frac{\psi(\xi)J_0(\xi\rho)}{\sinh \xi\,\delta}\,d\xi \tag{20}$$

by which to determine the displacement field.

Further, if we substitute from Eq. (18) into Eq. (14) and put $\zeta = 0$, we find that

$$[\sigma_z]_{z=0} = -\frac{E(1-\nu)\epsilon}{(1+\nu)(1-2\nu)\,d}\int_0^\infty \psi(\xi)K(\xi\,\delta)J_0(\xi\rho)\,d\xi, \tag{21}$$

where

$$K(\xi\,\delta) \equiv \xi\,\delta[1 + H(\xi\,\delta)] = \xi\,\delta\cdot\frac{\xi\,\delta + \sinh \xi\,\delta\cosh \xi\,\delta}{\sinh^2 \xi\,\delta}. \tag{22}$$

If we substitute the conditions (3) and (5) into Eqs. (20) and (21) we find that the function $\psi(\xi)$ must satisfy the dual integral equations

$$\int_0^\infty \xi[1 + H(\xi\,\delta)]\psi(\xi)J_0(\xi\rho)\,d\xi = \delta^{-1}, \qquad 0 \le \rho \le 1, \tag{23}$$

$$\int_0^\infty \psi(\xi)J_0(\xi\rho)\,d\xi = 0, \qquad \rho > 1. \tag{24}$$

The second of the two boundary value problems can be treated in a similar way. If we substitute from Eq. (16) into Eq. (14) and make use of the condition (10) we find that, in this case, the relation between $A(\xi)$ and $B(\xi)$ is now

$$(\xi^2\,\delta^2 - 2\nu\sinh^2 \xi\,\delta)A + (\xi\,\delta + \sinh \xi\,\delta\cosh \xi\,\delta)B = 0$$

which may be written in the form

$$A(\xi) = \frac{(1-2\nu)(1+\nu)F}{E}\chi(\xi), \qquad B(\xi) = -\frac{(1-2\nu)(1+\nu)F}{E}\cdot\frac{\xi^2\,\delta^2 - 2\nu\sinh^2 \xi\,\delta}{\xi\,\delta + \sinh \xi\,\delta\cosh \xi\,\delta}\chi(\xi).$$

The dual integral equations to be satisfied by $\chi(\xi)$ are then

$$\int_0^\infty \xi M(\xi\,\delta)\chi(\xi)J_0(\xi\rho)\,d\xi = 1, \qquad 0 \le \rho \le 1, \tag{25}$$

$$\int_0^\infty \chi(\xi)J_0(\xi\rho)\,d\xi = 0, \qquad \rho \ge 1, \tag{26}$$

where

$$M(\xi\,\delta) = \frac{\sinh^2 \xi\,\delta - \xi^2\,\delta^2}{\xi\,\delta + \cosh \xi\,\delta\sinh \xi\,\delta}. \tag{27}$$

5. Solution of the dual integral equations. In this section we shall derive the solution of the pair of dual integral equations (23), (24) using the method of Lebedev and Uflyand

[8]. Equations (25), (26) can be solved by a similar procedure. The essence of the Lebedev-Uflyand method is that it reduces the solution of a pair of dual integral equations to a Fredholm integral equation of the second kind which can then be solved by standard methods [10].

Since we are interested in values of δ much greater than 1 we write the function $H(\xi\delta)$ in the form

$$H(\xi\ \delta) = 2\sum_{m=1}^{\infty}(1 + 2m\xi\ \delta)e^{-2m\xi\delta}. \tag{28}$$

Following Lebedev and Uflyand we write

$$\psi(\xi) = \int_0^1 \phi(t)\sin(\xi t)\,dt, \qquad \phi(0) = 0. \tag{29}$$

It is easily shown that the equation (24) is satisfied identically by this representation of $\psi(\xi)$. Substituting from Eq. (29) into (23) we find that

$$\int_0^\infty J_0(\xi\rho)\,d\xi \int_0^1 \xi\phi(t)\sin(\xi t)\,dt + \int_0^\infty \xi H(\xi\ \delta)J_0(\xi\rho)\,d\xi \cdot \int_0^1 \phi(t)\sin(\xi t)\,dt = \delta^{-1}, \qquad 0 \le \rho \le 1. \tag{30}$$

Since $\phi(0) = 0$ we see on integrating by parts that

$$\int_0^1 \xi\phi(t)\sin(\xi t)\,dt = -\phi(1)\cos\xi + \int_0^1 \phi'(t)\cos(\xi t)\,dt.$$

Now

$$\int_0^\infty J_0(\xi\rho)\cos(\xi t)\,dt = \begin{cases} 0, & 0 < \rho < t \\ (\rho^2 - t^2)^{-1/2}, & \rho > t, \end{cases} \tag{31}$$

so that, by a simple change in the orders of integration we find that the first term on the left side of Eq. (30) becomes

$$\int_0^\rho (\rho^2 - t^2)^{-1/2}\phi'(t)\,dt.$$

If in the second term on the left side of Eq. (30) we replace $J_0(\xi\rho)$ by its integral representation

$$(2/\pi)\int_0^\rho (\rho^2 - u^2)^{-1/2}\cos(u\xi)\,du,$$

we find that this term reduces to

$$\frac{2}{\pi}\int_0^\rho \frac{du}{\sqrt{(\rho^2 - u^2)}}\int_0^1 \tfrac{1}{2}[H^*(u + t) - H^*(u - t)]\phi(t)\,dt,$$

where

$$H^*(w) = \delta\int_0^\infty \xi H(\xi\ \delta)\sin(\xi w)\,d\xi. \tag{32}$$

We thus obtain the equation

$$\int_0^\rho (\rho^2 - t^2)^{-1/2}\phi'(t)\,dt + (2/\pi\,\delta)\int_0^\rho (\rho^2 - u^2)^{-1/2}\,du$$

$$\cdot\int_0^1 \phi(t)K_1(u, t)\,dt = \delta^{-1}, \qquad 0 \leq \rho \leq 1, \tag{33}$$

where $K_1(u, t) = \frac{1}{2}[H^*(u + t) - H^*(u - t)]$. We now let

$$\int_0^\rho (\rho^2 - t^2)^{-1/2}\phi'(t)\,dt = f(\rho); \tag{34}$$

then it is well-known ([10], p. 39) that

$$\phi'(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t v(t^2 - v^2)^{-1/2}f(v)\,dv$$

and since $\phi(0) = 0$ we may integrate to obtain the expression

$$\phi(t) = \frac{2}{\pi}\int_0^t \frac{vf(v)\,dv}{\sqrt{(t^2 - v^2)}}. \tag{35}$$

If we substitute this expression into Eq. (33), change the order of the integrations and write

$$K(\rho, v) = v\int_v^1 (t^2 - v^2)^{-1/2}\,dt\int_0^\rho (\rho^2 - u^2)^{-1/2}\,\delta^2K_1(u, t)\,du, \qquad 0 < \rho < 1 \tag{36}$$

we find that Eq. (33) reduces to the Fredholm equation

$$f(\rho) + 4\pi^{-2}\,\delta^{-3}\int_0^1 f(v)K(\rho, v)\,dv = \delta^{-1}, \qquad 0 < \rho < 1. \tag{37}$$

By the classical theory of these equations ([10], p. 49) we can obtain a solution of this equation as a power series in $(4\pi^{-2}\delta^{-3})$ provided that δ is sufficiently large for $\frac{1}{4}\pi^2\delta^3 > M$, where M is the bound on the kernel $K(\rho, v)$. In this way we obtain the solution

$$f(\rho) = \delta^{-1}\left\{1 - 4\pi^{-2}\,\delta^{-3}\int_0^1 K(\rho, v)\,dv + (4\pi^{-2}\,\delta^{-3})^2\int_0^1\int_0^1 K(\rho, v)K(v, w)\,dv\,dw + \cdots\right\}.$$

Since $K(\rho, v)$ is of order 0 in δ this means that

$$f(\rho) = \delta^{-1}\left\{1 - 4\pi^{-2}\,\delta^{-3}\int_0^1 K(\rho, v)\,dv\right\} + 0(\delta^{-7}). \tag{38}$$

To obtain the solution of Eq. (37) to this order we need only calculate

$$\int_0^1 K(\rho, v)\,dv = \int_0^\rho (\rho^2 - w^2)^{-1/2}\,dw\int_0^1 t[\delta^2K_1(w, t)]\,dt. \tag{39}$$

Using the expansion (28) for H we find from Eq. (32) that

$$H^*(u) = \tfrac{1}{2}\,\delta^{-2}\sum_{m=1}^{\infty} m^{-3}\{u[1 + (u/2m\,\delta)^2]^{-2} + u[1 + (u/2m\,\delta)^2]^{-3}[3 - (u/2m\,\delta)^2]\}$$

so that

$$\kappa(w) \equiv \int_0^1 t[\delta^2 K_1(w, t)]\, dt = \sum_{m=1}^{\infty} m^{-3}(I_1 - I_2 + I_3 - I_4), \tag{40}$$

where

$$I_1 = \tfrac{1}{4}\int_0^1 \frac{t(w+t)\,dt}{[1+\{(w+t)/2m\,\delta\}^2]^{\frac{5}{2}}} = m^3\,\delta^3[\tan^{-1}\{(w+1)/2m\,\delta\} - \tan^{-1}(w/2m\,\delta)] - \tfrac{1}{2}m^2\,\delta^2[1+\{(w+1)/2m\,\delta\}^2]^{-1},$$

$$I_2 = \tfrac{1}{4}\int_0^1 \frac{t(w-t)\,dt}{[1+\{(w-t)/2m\,\delta\}^2]^{\frac{5}{2}}} = m^3\,\delta^3[\tan^{-1}\{(w-1)/2m\,\delta\} - \tan^{-1}(w/2m\,\delta)] + \tfrac{1}{2}m^2\,\delta^2[1+\{(w-1)/2m\,\delta\}^2]^{-1},$$

$$I_3 = \tfrac{1}{4}\int_0^1 \frac{t(w+t)[3-\{(w+t)/2m\,\delta\}^2]\,dt}{[1+\{(w+t)/2m\,\delta\}^2]^3} = \tfrac{1}{8}[\{(w+2)(w+1)^2 + 4wm^2\,\delta^2\}\cdot\{1+[(w+1)/2m\,\delta]^2\}^{-2} - (w^3+4wm^2\,\delta^2)[1+w^2/4\,\delta^2m^2)^{-1}],$$

$$I_4 = \tfrac{1}{4}\int_0^1 \frac{t(w-t)[3-\{(w-t)/2m\,\delta\}^2]\,dt}{[1+\{(w-t)/2m\,\delta\}^2]^3} = \tfrac{1}{8}[\{(w-2)(w-1)^2 + 4wm^2\,\delta^2\}\cdot\{1+[(w-1)/2m\,\delta]^2\}^{-2} - (w^3+4wm^2\,\delta^2)(1+w^2/4\,\delta^2m^2)^{-1}].$$

Since the solution (38) is valid only to order δ^{-6} we replace I_1, I_2, I_3, I_4 by their expansions in powers of δ up to δ^{-2}. We find that

$$\kappa(w) = \frac{2}{3}\zeta(3) - \frac{3}{20}\,\delta^{-2}\,\zeta(5)(5w^2+2) + 0(\delta^{-4}), \tag{41}$$

where

$$\zeta(3) = \sum_{m=1}^{\infty} m^{-3} = 1.202, \qquad \zeta(5) \equiv \sum_{m=1}^{\infty} m^{-5} = 1.0369.$$

The numerical values of the zeta-functions $\zeta(3)$, $\zeta(5)$ are taken from [11]. Substituting from Eqs. (40) and (41) into Eq. (38) we obtain the expression

$$f(\rho) = \delta^{-1}[1 - 4\zeta(3)/(3\pi\,\delta^3) + 3\zeta(5)(5\rho^2+4)/(20\pi\,\delta^5) + 0(\delta^{-7})]. \tag{42}$$

From Eqs. (35), (29) we find that

$$\psi(\xi) = (2/\pi)\int_0^1 \sin\xi t\,dt \int_0^t \rho(t^2-\rho^2)^{-1/2} f(\rho)\,d\rho = -(2/\pi)[f_0\,D - \tfrac{2}{3}f_1\,D^3]\xi^{-1}\sin\xi + 0(\delta^{-7}), \tag{43}$$

where

$$f_0 = \delta^{-1}\{1 - 4\zeta(3)/(3\pi\,\delta^3) + 3\zeta(5)/(5\pi\,\delta^5)\}, \qquad f_1 = 3\zeta(5)/(4\pi\,\delta^5) \tag{44}$$

and D denotes the operator $d/d\xi$.

6. Shape of the deformed crack. We shall now proceed to find an approximate expression—correct to order δ^{-6}—for the displacement w on $z = 0$, $0 \le \rho \le 1$. Putting

$\zeta = 0$ in Eq. (20) we find that the normal component of the surface displacement is given by the equation

$$w_0(\rho) = \frac{2(1-\nu)^2}{1-2\nu}\epsilon\int_0^\infty \psi(\xi)J_0(\xi\rho)\,d\xi$$

which by Eq. (29) and the result

$$\int_0^\infty J_0(\xi\rho)\sin(\xi t)\,d\xi = \begin{cases} 0, & t < \rho; \\ (t^2-\rho^2)^{-1/2}, & t > \rho \end{cases}$$

we find that

$$w_0(\rho) = 2(1-\nu)^2(1-2\nu)^{-1}\epsilon\int_\rho^1 (t^2-\rho^2)^{1/2}\phi(t)\,dt$$

which, because of (35) can be written in the form

$$w_0(\rho) = 4(1-\nu)^2\epsilon/[\pi(1-2\nu)]\int_\rho^1 (t^2-\rho^2)^{-1/2}\,dt\int_0^t v(t^2-v^2)^{-1/2}f(v)\,dv.$$

Substituting for $f(v)$ from Eq. (42) and performing the integrations, we find that

$$w_0(\rho) = \frac{4(1-\nu)^2}{\pi(1-2\nu)}\left(\frac{\epsilon}{\delta}\right)(1-\rho^2)^{1/2}$$
$$\cdot\{1 - [4\zeta(3)]/(3\pi\,\delta^3) - [\zeta(5)(23+10\rho^2)]/(30\pi\,\delta^5) + 0(\delta^{-7})\}.$$

Using the numerical values of the zeta-functions we find that

$$w_0(\rho) = \{4(1-\nu)^2\epsilon/[\pi\,\delta(1-2\nu)]\}\cdot(1-\rho^2)^{1/2}k(\rho,\,\delta), \tag{45}$$

where "the correction term" $k(\rho, \delta)$ is defined by the equation

$$k(\rho,\,\delta) = 1 - 0.510\,\delta^{-3} - 0.011\,\delta^{-5}(23+10\rho^2) + 0(\delta^{-7}). \tag{46}$$

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ON A PROBLEM OF MINIMUM WEIGHT DESIGN*

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Summary. A problem of optimal design for perfectly plastic, isotropic structures is analyzed. It is shown that for such structures as plates or shells, an extremum of the volume, if it exists, is either a local maximum or a minimum.

1. Condition for an extremum of the volume. Consider a region R of space that is bounded by a regular surface S. On the part S_T of S, let the non-vanishing surface tractions T_i be prescribed, and on another part S_U of S, let the velocity u_i be required to vanish. It is assumed that S_T and S_u comprise the whole of the surface S. A rigid, perfectly plastic body B is to be designed to the following specifications: S_T and S_U form parts of the surface S_B of B; the remainder S_B' of S_B is to be free from surface tractions, while S_T is loaded in the prescribed manner and S_U is rigidly supported; the body B is to be contained in R and should reach its load carrying capacity under the prescribed surface tractions; it is to have the minimum volume possible under these conditions. This problem has been studied by Drucker and Shield [4, 5]†; we shall briefly discuss it to lay the foundation for subsequent work.

Consider a rigid, perfectly plastic body C that satisfies all conditions of the problem except that its volume V_c need not represent a minimum. As this body is supposed to reach its load carrying capacity under the prescribed surface tractions, there exists a stress field σ_{ij}^c that is statically admissible for these loads and does nowhere exceed the yield point. Moreover, there exists a kinematically admissible velocity field u_i^c the strain rates ϵ_{ij}^c of which are compatible with the stresses σ_{ij}^c and do not vanish identically. The principle of virtual velocities then furnishes the relation‡

$$\int \sigma_{ij}^c \epsilon_{ij}^c \, dV_c = \int T_i u_i^c \, dS_T \, . \tag{1}$$

Denoting the surface of this body by $S_c = S_T + S_u + S_c'$, we consider a modification of the stress-free part S_c' of this surface, such, that the altered body remains at the yield point under the same surface tractions T_i. If we move S_c' only outward (boundary E' in Fig. 1), the load-carrying capacity, in general, will increase (*cf.* [15]). So, the variation of S_c' should be performed in such a way that a part of S_c' is moved outside, the remainder inside of the body V_c (boundary E in Fig. 1). Further, we assume that there exists a stress field σ_{ij}^* defined throughout the volume $V^* = V_c + \delta V$ of the modified body that is statically admissible for the prescribed surface tractions and does nowhere exceed the yield limit. By the first fundamental theorem of limit analysis ([14], p. 40), it follows from this assumption that the prescribed surface tractions cannot exceed the load carrying capacity of the modified body. We further assume that the velocity field u_i^c is continued in a kinematically admissible manner into any volume that has been

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†Numbers in square brackets refer to the Bibliography at the end of the paper.

‡Body forces which do not alter the final conclusions, are not included here.

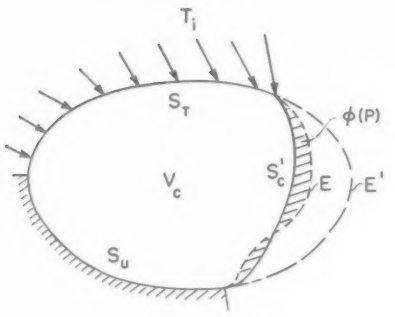

FIG. 1

added to V_c . It means that the field of strain rates ϵ^c_{ij} derived from u^c_i satisfies everywhere the incompressibility condition; moreover ϵ^c_{ij} are continuous functions within δV and on the surface S'_c . Obviously, in general, there are many such extensions beyond the volume V_c . If some additional kinematical constraints are imposed, the field of ϵ^c_{ij} has to be compatible with them and the number of such extensions may be reduced, even to one possible field. (e.g. plates, shells etc. subject to thickness variation). In the volume δV let σ^c_{ij} denote a stress field that corresponds to the extended strain rate field by the theory of the plastic potential ([14], p. 15) excluding rigid regions, where σ^c_{ij} is not defined. Setting $\sigma^*_{ij} = \sigma^c_{ij} + \delta\sigma_{ij}$ throughout the volume V^*, and applying the principle of virtual velocities to the stresses σ^*_{ij} and the strain rates ϵ^c_{ij} , we obtain

$$\int \sigma^c_{ij}\epsilon^c_{ij}\, dV_c + \int \sigma^c_{ij}\epsilon^c_{ij}\, d(\delta V) + \int \delta\sigma_{ij}\epsilon^c_{ij}\, dV^* = \int T_i u_i\, dS_T\,. \tag{2}$$

Here, the power of plastic dissipation $D = \sigma^c_{ij}\epsilon^c_{ij}$ is completely specified by the strain rates ϵ^c_{ij} ([14], p. 37).

Subtracting (1) from (2), we obtain:

$$\int D(\epsilon^c_{ij})\, d(\delta V) = -\int \delta\sigma_{ij}\epsilon^c_{ij}\, dV^*. \tag{3}$$

The following theorem may now be proved:* *If* $0 < D\ (\epsilon^c_{ij}) = \bar{D} = \text{const.}$ *on* S'_c *and* $D(\epsilon^c_{ij}) \geq \bar{D}$ *throughout* V_c , *and if the velocity field* u^c_i *can be so continued in a kinematically admissible manner beyond* S'_c *that the strain rates of this continuation satisfy* $D(\epsilon^c_{ij}) \leq \bar{D}$, *the body C has the minimum volume that can be obtained if the given loads are not to exceed the load carrying capacity.* Indeed, setting $D(\epsilon^c_{ij}) = \bar{D} + \Delta D$, we write (3) in the form:

$$\bar{D}\ \delta V = -\int \delta\sigma_{ij}\epsilon^c_{ij}\, dV^* - \int \Delta\ Dd(\delta V). \tag{4}$$

By the principle of the maximum specific power of dissipation ([14], p. 37), we have $\delta\sigma_{ij}\epsilon^c_{ij} \leq 0$, i.e. the first term on the right side of (4) is always positive and so is the second, since for points in the exterior of the body $\Delta D \leq 0$, $d(\delta V) > 0$, while $\Delta D \geq 0$, $d(\delta V) < 0$ for interior points. Thus, $\delta V > 0$ whenever S'_c is modified so that the pre-

*An analogous theorem has been stated in Ref. [5]; it is not clear however, which parts of the body were assumed to vary. If the part S_u may also undergo the variation, this theorem will no longer be true.

scribed surface tractions do not exceed the load carrying capacity of the modified body.

A proof of this theorem may also be provided by means of the second theorem of limit analysis*. Indeed, considering the modified body V^{**}, we assume that it has not yet reached or is just reaching its load carrying capacity. The kinematically admissible velocity field u_i^c cannot therefore be unstable ([14], p. 40) and we have

$$\int D(\epsilon_{ij}^c)\, dV^* \geq \int T_i u_i^c\, dS_T\,. \tag{5}$$

Setting $D = \bar{D} + \Delta D$ and comparing (1) and (5) one finds

$$\bar{D}\,\delta V \geq -\int \Delta D d(\delta V) \geq 0, \tag{6}$$

which completes the proof.

A special case of this theorem arises when $D(\epsilon_{ij}^c) = \bar{D} = \text{const.}$ inside and outside of the body V_c. Then $\Delta D = 0$ and the right side of Eq. (4) contains only the first term. This case occurs, for instance, in discs, membranes or sandwich plates. If we consider a disc that is subjected to prescribed surface tractions T_i on all of the cylindrical surface S_T forming its "edge", only the lateral surfaces S_c', that is the thickness of the disc, may be varied to minimize the volume. Since stress and strain rate are constant across the thickness of the disc, the constancy of $\bar{D}$ on the lateral surfaces implies that $D(\epsilon_{ij}^c) = \bar{D} = \text{const}$ everywhere. This design criterion for discs has been given in [4] and [5].

Difficulties arise, however, when one deals with structures, such as plates and shells, for which $D(\epsilon_{ij}^c)$ is greater in the exterior than in the interior of V_c. The theorem proved above then is no longer applicable. Nevertheless, we shall try to choose the surface S_c' so as to render $D(\epsilon_{ij}^c) = \bar{D} = \text{const.}$ on S_c'. We shall see that this type of design has certain advantages.

An infinitesimal modification of the surface S_c' can be specified in terms of the normal displacement $\eta\varphi(P)$ of the typical point P. Here, η is an infinitesimal constant, and $\varphi(P)$ a continuous function of the position of the point P on S_c'. If $\delta\sigma_{ij}$ is of the first order* (i.e. of the same order as η), then $\delta\sigma_{ij}\epsilon_{ij}^c$ is negative and of the second or lower order according to whether the body V^* is everywhere at the yield point or not. Thus, the first term on the right side of Eq. (4) is positive and of the second or lower order, while the second term is negative because now $\Delta D \geq 0$, $d(\delta V) > 0$ in the exterior and $\Delta D \leq 0$, $d(\delta V) < 0$ in the interior of V_c. This second term is likewise of the second order, since ΔD is a first order quantity in the vicinity of S_c'. Thus, Eq. (4) implies the following: *a body for which $D(\epsilon_{ij}^c) = \bar{D} = \text{const. on } S_c'$ attains either a local minimum or a local maximum*, depending on the sign of the expression on the right side of Eq. (4). This sign must therefore be studied in each individual case. It should be noted that, whereas the minimum is not necessarily analytical, the maximum must be analytical.

It is interesting to note, that in the case considered last the second theorem of limit

*The author is indebted to Professor W. Prager for this remark. In fact, the proof based on limit analysis theorems has been presented in Ref. [5].

**The conditions that the function $\varphi(P)$ has to satisfy in order to ensure $\delta\sigma_{ij}$ to be of the first order are not yet known in the general case. In the following, this problem will be discussed in detail for a circular plate, symmetrically loaded.

analysis does not provide sufficient information on the sign of δV. Indeed, the inequality (6) now shows only that δV is greater than a negative quantity of the second order. From this fact, no conclusions can be drawn regarding the sign or order of δV. Equation (4), however, shows that we have an extremum of the volume.

A somewhat different formulation of the problem of optimum design arises for bodies that are required to have a plane, axis, or center, of symmetry. Then any modification of a part of the surface S_c' entails the corresponding modification of the symmetrically located part. If the prescribed surface tractions T_i and the kinematically admissible velocity field u_i^c need not have the symmetry properties of the body, the condition of constant $D(\epsilon_{ij}^c)$ on S_c' must be replaced by the condition

$$D^{(1)}(\epsilon_{ij}^c) + D^{(2)}(\epsilon_{ij}^c) = \bar{D} = \text{const.}, \tag{7}$$

where $D^{(1)}$ and $D^{(2)}$ denote the values of $D(\epsilon_{ij}^c)$ at a typical pair of symmetrically situated points of S_c' .

2. Design of plates. Consider a solid plate of given plane-form that is subjected to prescribed transverse loads and supported in a given manner. A particular design of this plate is specified by the distribution of the plate thickness over the plane-form, and the discussion will be restricted to designs that reach their load carrying capacity under the given loads. The problem of finding the design of minimum volume has been thoroughly investigated for circular plates under rotationally symmetric loading [1, 3, 6, 7] and some results have been obtained for other shapes or loads [2, 8, 9]. Drucker and Shield (4, 5) have established conditions under which the volume of a plate is stationary with respect to neighboring designs. As we shall see, these conditions may lead to a local maximum as well as a local minimum.

Let $2h_c$ be the variable thickness of a design for which the given loads represent the load carrying capacity. Denote the rates of extension and shear at the lower surface of the plate by ϵ_{ij}^0 , $(i, j = 1, 2)$, and assume that the specific power of dissipation has the constant value $\bar{D}$ at this surface:

$$D(\epsilon_{ij}^0) = \bar{D}. \tag{8}$$

As is customary in plate theory, we shall assume the strain rates to vary linearly over the thickness of the plate. At the distance x_3 below the median plane of the plate, we therefore have the strain rates

$$\epsilon_{ij}^c = (x_3/h_c)\epsilon_{ij}^0 , \qquad (i, j = 1, 2). \tag{9}$$

Since the dissipation function is homogeneous of the order one in the strain rates and does not change its value when the signs of all strain rate components are reversed, we have by (8) and (9)

$$D(\epsilon_{ij}^c) = (|\, x_3 \,|/h_c)\, \bar{D}. \tag{10}$$

Equation (9) also represents the kinematically admissible extension of the strain rate field beyond the lateral surfaces of the plate that will be used in the following.

The state of stress at a typical point of the fully plastic plate will be treated as plane, and the notation σ_{ij}^0 , $(i, j = 1, 2)$, will be used for the stresses in the lower half of the plate, those in the upper half having the same intensities but opposite signs. The same stresses are compatible with the extension (9) of the strain rate field.

If the variation of the plate thickness is denoted by $2\delta h$, and the element of area of the median plane by dA, the terms on the right side of (4) can be evaluated as follows:

$$\int \delta\sigma_{ij}\epsilon^c_{ij}\, dV^* = 2\int dA \int_0^{h_c+\delta h} \delta\sigma_{ij}\epsilon^0_{ij}(x_3/h_c)\, dx_3 = \int \delta\sigma_{ij}\epsilon^0_{ij}[(h_c + \delta h)^2/h_c]\, dA, \tag{11}$$

$$\int \Delta\, Dd(\delta V) = 2\int dA \int_{h_c}^{h_c+\delta h} [(x_3/h_c) - 1]\, \bar{D}\, dx_3 = \bar{D}\int (\delta h^2/h_c)\, dA. \tag{12}$$

Equation (4) therefore furnishes

$$\delta V = -\int (\delta h^2/h_c)\, dA - 1/\bar{D}\int \delta\sigma_{ij}\epsilon^0_{ij}[(h_c + \delta h)^2/h_c]\, dA. \tag{13}$$

If the yield locus in the space with the rectangular Cartesian coordinates σ_{11} , σ_{22} , $\sigma_{12}(2)^{\frac{1}{2}}$ has continuously turning tangent plane, $\delta\sigma_{ij}\epsilon^0_{ij}$ is negative and of the second order. The two terms on the right side of (13) are therefore of the second order and of opposite signs. This indicates that the volume of the original plate is stationary in comparison to neighboring plates that reach their load carrying capacity under the prescribed surface tractions, but no general statement can be made regarding the character of this stationary value (maximum, minimum, or saddle point).

If, on the other hand, the yield locus has a plane face and the states of stress in the lower half of the plane are represented by interior points of this face, for the original as well as the modified plate, then $\delta\sigma_{ij}\epsilon^0_{ij} = 0$. The right-hand side of (13) therefore is negative and of the second order, so that the original plate represents an analytical local *maximum* of the volume. The same remark applies if the yield locus contains a straight segment and the states of stress in the lower half of the plate are represented by interior points of this segment for both the original and the modified plate.

Finally, if the state of stress throughout the lower half of the plate is represented by points of an edge of the yield locus, $\delta\epsilon_{ij}\epsilon^0_{ij}$ will as a rule be negative and of the first order, so that the first term on the right side of (13) can be neglected. The original plate then represents a non-analytical local minimum of the volume. This statement, however, must be modified, if for each considered stress point σ^0_{11} , σ^0_{22} , $\sigma^0_{12}(2)^{\frac{1}{2}}$ on an edge of the yield locus, the vector with the components ϵ^0_{11} , ϵ^0_{22} , $\epsilon^0_{12}(2)^{\frac{1}{2}}$ happens to be normal to one of the faces intersecting in this edge. If then the corresponding stress point for the modified plate lies on this face, $\delta\sigma_{ij}\epsilon_{ij}$ will be negative and of the second order; if, however, this stress point lies on the other face, $\delta\sigma_{ij}\epsilon_{ij}$ will be negative and of the first order. In the first case, we have a stationary value, in the second a local minimum of the volume.

Equation (13) may also be rewritten in the following form:

$$\delta V = -\int (\delta h^2/h_c)\, dA - \frac{1}{\bar{D}}\int \delta M'_{ij}\kappa^c_{ij}[(h_c + \delta h)/h_c]^2\, dA, \tag{14}$$

where $\delta M'_{ij} = \delta\sigma_{ij}h_c^2$ denotes the variation of bending moments due to the change of the stress components only and κ^c_{ij} is the rate of curvature. Figure 2 represents the

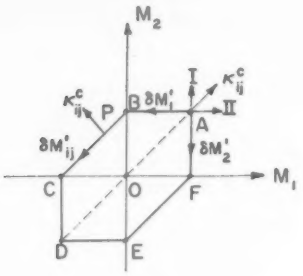

FIG. 2

Tresca yield condition referred to the principal bending moments. In view of the above, we conclude that for sides of the hexagon the optimal solution represents a maximum of the volume, while for the corners of the hexagon the plate will have either a minimal volume or its volume will correspond to a saddle point. The last case occurs, for instance, when κ_{ij}^c coincides with one of the normals to the adjacent sides, say I at the corner A; the solution obtained will then represent a maximal volume for all states AB and will be a minimum with respect to all states AF, i.e. it will correspond to a saddle point.

Example 1. Consider first a simply supported circular plate of the radius a under the uniformly distributed load q, and adopt the Tresca yield condition. On account of the rotational symmetry, the principal bending moments are the radial moment M_r and the circumferential moment M_φ ; the associated strain rates are the radial and circumferential rates of curvature κ_r and κ_φ .

Hopkins and Prager [1] treated this problem of optimum design assuming that the entire plate is at states of stress corresponding to the corner A of the hexagon (Fig. 2). For the plastic moment $M_0 = \sigma_0 h_c^2$ (σ_0 being yield stress in simple tension) they obtained the expressions

$$M_0 = \frac{q}{4}(a^2 - r^2), \qquad h_c = \tfrac{1}{2}(q/\sigma_0)^{1/2}(a^2 - r^2)^{1/2}, \qquad \text{(Fig. 3a)}. \tag{15}$$

The corresponding distribution of the rate of deflection w can be determined as follows. Starting from the condition $\bar{D} = \text{const.}$ we write

$$\bar{D} = \frac{M_r\kappa_r + M_\varphi\kappa_\varphi}{h_c} = \alpha, \quad \text{or} \quad \frac{1}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2} = -\frac{\bar{\alpha}}{(a^2 - r^2)^{1/2}}, \tag{16}$$

where $\alpha = \text{const.}$ and $\bar{\alpha} = 2\alpha/(g\sigma)^{1/2}$. Integrating Eq. (16), one obtains

$$w = \bar{\alpha}\left[-a \ln \frac{r}{a} + (a^2 - r^2)^{1/2} - a \ln \{[a + (a^2 - r^2)^{1/2}]/r\}\right].$$

It is easily verified that $\kappa_\varphi \geq 0$ and $\kappa_r \geq 0$ within the plate, i.e. the solution really corresponds to the corner A.

For the side AB we have $\kappa_r = 0$, i.e. $d^2w/dr^2 = 0$ and $w = A(r - a)$, hence $\kappa_\varphi = -1/r\ dw/dr = A/r$. The constancy of $\bar{D}$ yields then

$$M_\varphi\kappa_\varphi/h_c = \text{const.} \quad \text{or} \quad h_c = K_1 r, \qquad \text{(Fig. 3b)}, \tag{17}$$

FIG. 3

where K_1 is an arbitrary constant. If the side BC is considered, then $\kappa_r = -\kappa_\varphi$, $w = A \ln a/r$, and the condition $\bar{D} = \text{const.}$ leads to the solution $h_c = K_2 r^2$. Since for the states represented by CD a plate cannot deform, there is no solution for this region. For the side AB we have

$$M_\varphi = \sigma_0 K_1^2 r^2 = Cr^2, \qquad M_r = \left(C - \frac{q}{2}\right)\frac{r^2}{3}.$$

Since $M_r = 0$ for $r = a$, $C = q/2$, $M_r = 0$ within the plate, thus the stress state is represented by the point B. The volume of this plate is $V = 0.946\,\pi(q/\sigma_0)^{\frac{1}{2}}\,a^3$ and is greater than that of a plate of constant thickness ($V = 0.815\,\pi(q/\sigma_0)^{\frac{1}{2}}\,a^3$). On the other hand it may be shown that no extremal solution exists for the state BC; thus plates corresponding to BC will have a greater volume than those corresponding to B, which therefore represents a saddle point.

An exact proof of the existence of a minimum for corners and a maximum for sides of the hexagon has been given in Ref. [7] in a somewhat different way.

Example 2. Consider a circular plate symmetrically loaded and made of a material that obeys Mises' yield condition. We then have

$$M_r^2 - M_r M_\varphi + M_\varphi^2 = M_0^2, \qquad (M_r + \delta M_r')^2 - (M_r + \delta M_r')(M_\varphi + \delta M_\varphi') + (M_\varphi + \delta M_\varphi')^2 = M_0^2, \tag{18}$$

$$\kappa_r = \lambda(2M_r - M_\varphi), \qquad \kappa_\varphi = \lambda(2M_\varphi - M_r), \qquad \text{(see [14])}, \tag{19}$$

and hence

$$\delta M_r' \kappa_r + \delta M_\varphi' \kappa_\varphi = \lambda[\delta M_r'(2M_r - M_\varphi) + \delta M_\varphi'(2M_\varphi - M_r)] = -\lambda(\delta M_r'^2 - \delta M_r'\,\delta M_\varphi' + \delta M_\varphi'^2). \tag{20}$$

The quantities $\delta M_r'$ and $\delta M_\varphi'$ can be determined from the yield condition (18) and the equation of equilibrium. Writing

$$\frac{d}{dr}(r\,\delta M_r) - \delta M_\varphi = 0, \qquad \delta M_r = 2M_r\frac{\delta h}{h_c} + \delta M_r', \qquad \delta M_\varphi = 2M_\varphi\frac{\delta h}{h_c} + \delta M_\varphi', \tag{21}$$

we have

$$\frac{d}{dr}(r\,\delta M_r') - \delta M_\varphi' = -2\frac{d}{dr}\left(rM_r\frac{\delta h}{h_c}\right) + 2M_\varphi\frac{\delta h}{h_c}. \tag{22}$$

Equations (18) and (22) determine $\delta M_r'$ and $\delta M_\varphi'$. From these equations it follows that $\delta M_r'$ and $\delta M_\varphi'$ will be of the same order as δh, provided $d(\delta h)/dr$ will be also of the same order. In view of (20) $\delta M_{ij}'\,\kappa_{ij}^c$ is therefore of the second order and an extremum of the volume is obtained.

Since the expression (14) represents the Weierstrass' function for the volume of a plate, we can use $\delta M_r'$ as the variation rather than δh; note that $\delta M_r'$ vanishes at the edge. We can then apply the known criterion for a minimum; setting $M_r' = dM_r/dr$, we have

$$\frac{\partial^2 F}{\partial M_r'^2} = \frac{\partial^2[(M_r^2 - M_r M_\varphi + M_\varphi^2)^{1/2} r]}{\partial M_r'^2} \geq 0, \tag{23}$$

provided the Jacobi condition is satisfied. On performing the calculations, we obtain

$$5M_r^2 + 4M_r M_\varphi - 4M_\varphi^2 \geq 0, \tag{24}$$

and hence

$$-0.725 \leq M_\varphi/M_r \leq 1.725$$

The dashed region shown in Fig. 4 represents the inequalities (25). If the solution

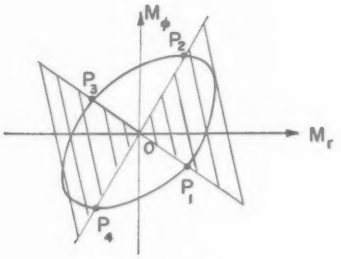

FIG. 4

lies within this region, it will correspond to the minimum of the volume. A problem of optimal design using Mises' yield condition has been studied by Freiberger and Tekinalp [3]. It can be easily checked, that the solution obtained by them is contained within the angle P_1OP_2 (Fig. 4). It is possible, however, that in the region P_2OP_3 there is another solution, for which the volume attains its maximum or corresponds to a saddle point.

If an extremal solution exists, which lies in the region P_1OP_2 then there is no other solution in this region. Indeed, in the transition from one minimal solution to the other, we must pass the maximum, i.e. come into the region P_2OP_3 and then move back. This implies a contradiction, since we would have passed the same minimal state twice. The other minimal solution, if it exists, must therefore lie in the region OP_3P_4. Furthermore, since there is no upper bound for the volume, if a maximum exists at all, some other minimal solution must also exist, unless we have a saddle point.

3. Design of shells. Assuming that the thickness of a shell is to be altered symmetrically with respect to the median surface, we cannot obtain $\bar{D} = \text{const.}$ on both surfaces. We should therefore apply the condition (7). Performing calculations similar to those presented for plates, we obtain the following results. When

$$\Delta = \sigma_{ij}^{+}(\epsilon_{ij}^c + \kappa_{ij}^c h_c) + \sigma_{ij}^{-}(\epsilon_{ij}^c - \kappa_{ij}^c h_c) = \text{const.} \tag{26}$$

at every point of a shell, we have (neglecting orders higher than second)

$$\Delta\, \delta V = -\frac{1}{2}\int_A \kappa_{ij}^c(\sigma_{ij}^{+} - \sigma_{ij}^{-})\, \delta h^2\, dA - \int_A \left[\int_{-h_c}^{+h_c} \delta\sigma_{ij}(\epsilon_{ij}^c + \kappa_{ij}^c z)\, dz\right] dA, \tag{27}$$

where σ_{ij}^{+} , σ_{ij}^{-} denote the stresses at the two surfaces and ϵ_{ij}^{c} , κ_{ij}^{c} are the strain rates and rates of curvature of the median surface. The first term in (27) is always positive, and the second, negative. Similarly as before, if $\delta\sigma_{ij} \sim \delta h$, we conclude that condition (26) assures an extremum of the volume. Examples may also be provided which show that in some cases the extremal solutions are maxima.

Problems of optimal design for shells have been studied in Refs. [10], [11], [12], [13]. In Ref. [12], the criterion (26) has been established as sufficient for a minimum of the volume, which is generally true for sandwich shells.

If we assume that the thickness of the shell may be changed on one surface only, while the other surface is kept unaltered, we must take into consideration that the median surface is changed; the problem therefore requires separate treatment. If, for instance, $\bar{D}$ = const. on the surface that is changed, is smaller than $\bar{D}$ on the other surface, our shell will attain an absolute minimum according to the theorem proved above.

Acknowledgement. This work was carried out while the author was a Post-Doctoral Fellow at the Brown University, Division of Applied Mathematics, Providence, R. I. The author wishes to express his gratitude to Professor W. Prager for his assistance during preparation of this paper and to Professors D. C. Drucker and R. T. Shield for their helpful discussion.

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BOOK REVIEWS

(*Continued from p. 110*)

Theoretical physics in the twentieth century. Edited by M. Fierz and V. F. Weisskopf. Interscience Publishers, New York and London, 1960. x + 328 pp. $10.00.

Pauli was considered by many physicists to have a deeper and keener insight into the fundamental problems of physics than anyone else in the past 50 years. This memorial volume to Wolfgang Pauli supplies the evidence for such a claim. There seems to be practically no field in physics to which Pauli has not contributed, and this account of his contributions written by experts makes extremely interesting reading.

ROHN TRUELL

Special relativity. By W. Rindler. Interscience Publishers, Inc., New York, 1960. x + 186 pp. $2.25.

This is intended to be a concise introduction to the subject of special relatively and it is. It is a text that can be recommended for seniors and first year graduate students. The reviewer has found the problems and exercises from the first four chapters useful for seniors studying wave phenomena and radiation physics. The book assumes only that the reader is familiar with calculus and vector methods. Tensor theory is developed, to the extent needed, in an appendix.

ROHN TRUELL

Annual review in automatic programming. Volume I: Papers read at the Working Conference on Automatic Programming of Digital Computers held at Brighton, 1-3 April 1959. Edited by R. Goodman. Pergamon Press, New York, Oxford, London, Paris, 1960. xi + 300 pp. $10.00.

Aside from papers on automatic coding systems for specific computers (mostly British) there are the following papers of more general character: Future trends in automatic programming (A. E. Glennie)—Some problems of a universal autocode (K. A. Redish)—The application of formula translation to automatic coding of ordinary differential equations (J. P. Cleave)—The philosophy of Programming (S. Gill)—Automatic programming and business applications (G. Cushing)—Auto-programming for numerically controlled machine tools (J. E. Meggitt).

Continuous geometry. By John von Neumann. Foreword by Israel Halperin. Princeton University Press, New Jersey, 1960. xi + 299 pp. $7.50.

This is a carefully edited and corrected reprint of lecture notes multigraphed in 1935–7; it provides a definitive technical account of a fascinating generalization of projective geometry, invented by von Neumann. This generalization has applications to operators on Hilbert space, and hence it sheds some light on the foundations of quantum mechanics.

No discussion of these applications is given in this book. However, as a display of brilliant algebraic techniques and incisive axiomatic analysis, by one of the most versatile pure *and* applied mathematicians of this century, the book is still extremely stimulating. In particular, it is one of the most penetrating studies available of the abstract algebra of subspaces of function space under intersection and union, and of the related algebra of semisimple and "regular" rings.

GARRETT BIRKHOFF

(*Continued on p. 142*)

HYDRODYNAMIC INERTIA COEFFICIENTS FOR A SLENDER BODY WITH A SAIL*

By

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Introduction. In the course of studying the hydrodynamics of a submerged submarine it becomes necessary to calculate the hydrodynamic inertia coefficients of a body whose cross-section consists of a circle with one sail (fin). The simpler inertia coefficients for this shape appear in the open literature, although somewhat obscured by the presence of what, for this problem, would be considered extraneous fins. The inertia coefficient which gives rise to rolling moment due to roll velocity is not in the literature, and it is principally with the calculation of this coefficient that the present paper is concerned.

Analysis. To determine the forces and moments on a rigid slender body moving through an infinite incompressible fluid where the motion of the fluid is irrotational and at rest far from the body, requires the evaluation of the six components of the symmetric inertia tensor of the cross section of the body

$$m_{jk} = m_{kj} = -\rho \oint \phi_j \frac{\partial \phi_k}{\partial n} ds \qquad j, k = 1, 2, 3. \tag{1}$$

Refer to Lamb [1], Munk [2], Jones [3], Spreiter [4] and Bryson [5].

The cross section of the configuration is mapped conformally onto a circle in order to determine the velocity potentials for the low-speed two-dimensional motion of the cross section. The inertia coefficients may then be determined from Eq. (1).

Consider a slender body of revolution of radius $a(x_3)$, bearing a sail of height $t(x_3) - a(x_3)$, as shown in cross section in Fig. 1. Because of the plane of symmetry of the cross section, we see that:

$$a) \quad m_{23} = 0. \tag{2}$$

That is, a motion in the x_2 direction cannot produce a rolling moment. In addition,

$$b) \quad m_{12} = 0. \tag{3}$$

That is, a motion in the x_1 direction cannot produce a force in the x_2 direction.

The sail does not affect the inertia coefficient m_{22} since it lies along the streamlines of the flow. Therefore, the case of no sail suffices to obtain m_{22} . The result is

$$\frac{m_{22}}{\rho \pi a^2} = 1. \tag{4}$$

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The expression for m_{11} may be taken directly from Summers [6] or Bryson [7]. The result is

$$\frac{m_{11}}{\rho\pi a^2} = \left\{1 + \frac{a^2}{4t^2}\left(\frac{t}{a} - 1\right)^2\left[\left(\frac{t}{a}\right)^2 + 6\left(\frac{t}{a}\right) + 1\right]\right\}. \tag{5}$$

Equation (5) has been plotted in Fig. 2.

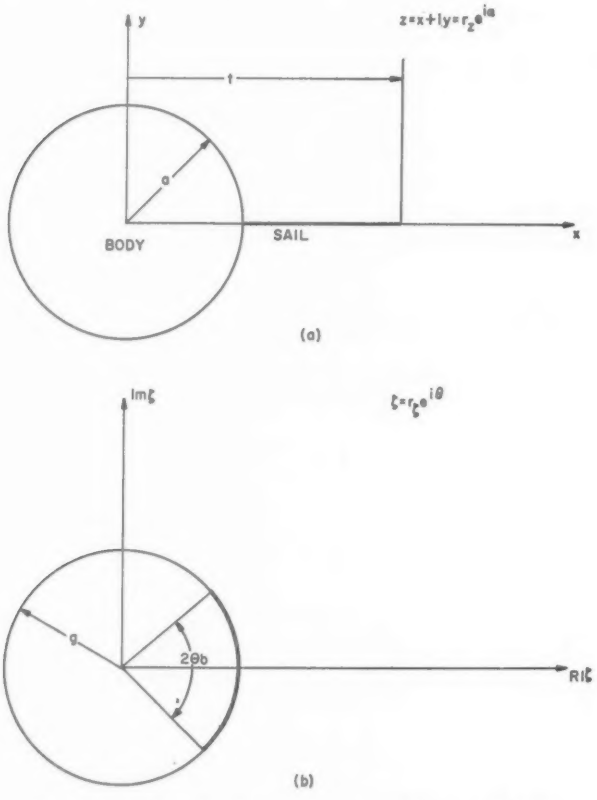

FIG. 1. Mapping of a circle with a sail onto a circle (Eq. 9)

The expression for m_{13} may be taken directly from Bryson [7]. The result is*

$$\frac{m_{13}}{\rho\pi a^3} = \frac{(t/a - 1)^2}{\pi}\left[\frac{(2\beta_0 + \pi)}{16}\frac{(t/a + 1)^4}{(t/a)^3} + \frac{(t/a - 1)}{4(t/a)^{1/2}}\left(1 + \frac{14}{3}\frac{1}{t/a} + \frac{1}{(t/a)^2}\right)\right], \tag{6}$$

where

$$\beta_0 = \frac{\theta_0}{2} = \sin^{-1}\left(\frac{t/a - 1}{t/a + 1}\right).$$

Equation (6) has also been plotted in Fig. 2.

*An error in sign has been corrected which makes m_{13} positive.

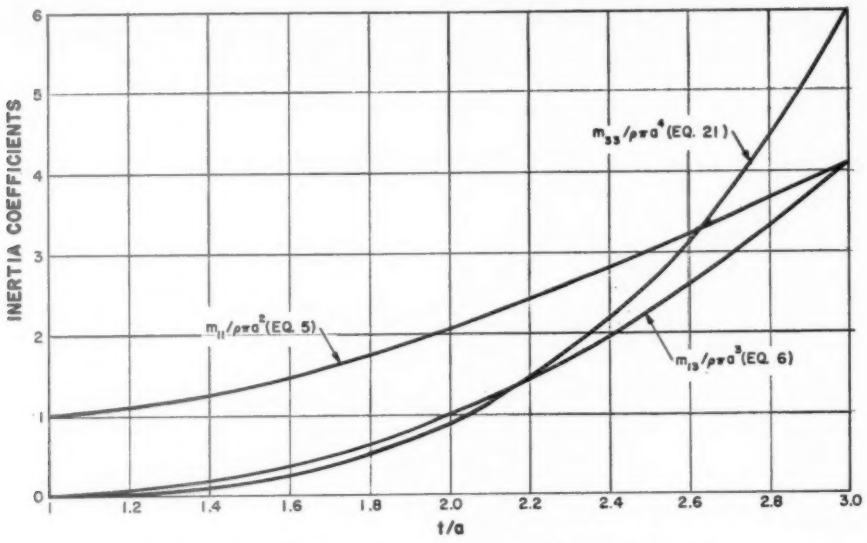

FIG. 2. Inertia coefficients as functions of sail length-radius ratio

The apparent additional moment of inertia, m_{33} , for a body of revolution with a sail is not given in the literature and will now be obtained.

It is required to find a solution, $\phi_3(x, y)$, to the two-dimensional Laplace equation in the $z = x + iy$ plane satisfying the boundary conditions

on the sail

$$\frac{\partial \phi_3}{\partial y}(x, y)\,|_{y=0} = x \qquad a < x < t, \tag{7}$$

on the body

$$\frac{\partial \phi_3}{\partial r_z}(r_z \cos\alpha, r_z \sin\alpha)\,|_{r_z=a} = 0, \tag{8}$$

where (x, y) are a set of right handed, cartesian coordinates and the longitudinal dimension is measured positive downstream from the body nose. Note that we have assumed unit angular velocity about this axis.

The conformal transformation

$$z\left(1 + \frac{a}{z}\right)^2 - 2g = \zeta + \frac{g^2}{\zeta}, \tag{9}$$

where

$$g = \frac{t}{4}\left(1 + \frac{a}{t}\right)^2 \tag{10}$$

maps the cross section of the body in the $z(= x + iy)$ plane (Fig. 1) on the circle $|\zeta| = g$ in the ζ plane, the sail appearing as an arc extending from $-\theta_0$ to θ_0 , where

$$\cos\theta_0 = \frac{2a}{g} - 1 = \frac{6a - t - (a^2/t)}{2a + t + (a^2/t)}. \tag{11}$$

On the sail the mapping function given by Eq. (9) and (10) reduces to

$$x = [g(1 + \cos\theta) - a] + \{g(1 + \cos\theta)\cdot[g(1 + \cos\theta) - 2a]\}^{1/2}. \tag{12}$$

The boundary conditions (7) and (8) transform to

$$\frac{\partial\phi_3}{\partial r_\zeta}(r_\zeta\ ,\ \theta)\ |_{r_\zeta=g} = \frac{2x\sin\theta}{1 - (a^2/x^2)} \qquad -\theta_0 < \theta < \theta_0\ , \tag{13}$$

$$\frac{\partial\phi_3}{\partial r_\zeta}(r_\zeta\ ,\ \theta)\ |_{r_\zeta=g} = 0 \qquad |\ \theta\ | > \theta_0\ . \tag{14}$$

The problem has thus been reduced to an exterior Neumann problem for a circle and the solution for ϕ_3 is given by

$$\phi_3(r_\zeta\ ,\ \theta) = \frac{g}{2\pi}\int_0^{2\pi}\frac{\partial\phi}{\partial r_\zeta}\bigg|_{r_\zeta=g}\ln\left|\ 1 - \frac{2g}{r_\zeta}\cos(\theta - \psi) + \left(\frac{g}{r_\zeta}\right)^2\ \right| d\psi. \tag{15}$$

On the circle $\zeta = ge^{i\theta}$ the solution for the potential ϕ_3 is given by

$$\phi_3(g,\ \theta) = \frac{g}{\pi}\int_0^{2\pi}\frac{\partial\phi}{\partial r_\zeta}\bigg|_{r_\zeta=g}\ln\left|\ 2\sin\frac{\theta - \psi}{2}\ \right| d\psi. \tag{16}$$

All integrals are to be evaluated in the Cauchy sense. Equations (13) and (14) are substituted into (16), and x is eliminated by virtue of Eq. (12). Then, after an integration by parts to eliminate the logarithm there is obtained:

$$\begin{aligned}\phi_3(g,\ \theta) = -\frac{g^2}{\pi}\sin\theta\int_0^{\theta_0}\Bigg\{&\cos\frac{\psi}{2}\left(2\cos^2\frac{\psi}{2} - \cos^2\frac{\theta_0}{2}\right)\left(\cos^2\frac{\psi}{2} - \cos^2\frac{\theta_0}{2}\right)^{1/2}\\ &+ 2\cos^2\frac{\psi}{2}\left(\cos^2\frac{\psi}{2} - \cos^2\frac{\theta_0}{2}\right)\Bigg\}\left(\cos^2\frac{\psi}{2} - \cos^2\frac{\theta}{2}\right)^{-1} d\psi,\end{aligned} \tag{17}$$

where $\cos\theta_0$ is defined in Eq. (11). Carrying out the integration it is found that

$$\begin{aligned}\phi_3(g,\ \theta) = -\frac{g^2}{\pi}\sin\theta\Bigg\{&\sin\theta_0 + (\theta_0 + \pi)(1 - \cos\theta_0 + \cos\theta)\\ &+ \frac{(\cos\theta - \cos\theta_0)(1 + \cos\theta)}{\sin\theta}\ln\left|\frac{\tan\frac{\theta_0}{2} + \tan\frac{\theta}{2}}{\tan\frac{\theta_0}{2} - \tan\frac{\theta}{2}}\right|\Bigg\}.\end{aligned} \tag{18}$$

Now, from Eq. (1)

$$m_{33} = -\rho\oint\phi_3\frac{\partial\phi_3}{\partial n}\,ds \tag{19}$$

and on the body surface

$$\frac{\partial\phi_3}{\partial n}\,ds = -x\,dx - y\,dy. \tag{20}$$

But Eq. (20) equals zero on the body surface, $| z | = a$; hence, the integration need only be confined to the sail. The resulting integral is quite tedious. The result is

$$\frac{m_{33}}{\pi\rho a^4} = \frac{4}{\pi^2\eta^4}\left\{\beta_0(\pi + \beta_0)\left(\frac{9}{2} - 8\eta + 4\eta^2\right)\right.$$
$$+ \beta_0(9 - 10\eta)[\eta(1 - \eta)]^{1/2} + \frac{\pi}{2}(14 - 25\eta + 15\eta^2 - 5\eta^3)[\eta(1 - \eta)]^{1/2} \quad (21)$$
$$\left. + (1 - \eta)\left[\frac{\pi^2}{8}(9 - 7\eta + \eta^2 + \eta^3) + \eta\left(\frac{9}{2} - 2\eta - 2\eta^2\right)\right]\right\},$$

where

$$\eta = \frac{a}{g} = \frac{4t/a}{\left(1 + \frac{t}{a}\right)^2} = \cos^2\beta_0 \,. \quad (22)$$

Equation (21) has been plotted in Fig. 2.

As a check on this result, it will be noted that the value of m_{33} is given by Miles for the case of a body of revolution bearing two fins of equal length which are mounted 180 degrees apart [8]. For very small fins, the value of m_{33} given by Eq. (21) for the single-finned body should be one-half of the value given by Miles for the two-finned body, since, if the fins are small, they will not mutually interfere. Carrying out the limiting process for both cases bears out this relationship. An additional check may be made by proceeding to the limit as the body shrinks to zero. This case is given by Bryson [9], and it can be shown that Eq. (21) yields the correct limit.*

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*The authors are indebted to the referee for pointing out this limiting case.

BOOK REVIEWS

(*Continued from p. 136*)

Classical mechanics. By H. C. Corben and P. Stehle. John Wiley & Sons, Inc., New York, 1960. xi + 389 pp. $12.00.

The first edition of this widely used text was reviewed on p. 334 of volume 9 of this *Quarterly*. The present edition contains many changes, the most important of which are as follows. A chapter on time-dependent forces and non-conservative motion has been added containing sections on the inverted pendulum, rocket motion, atmospheric drag, the Poynting-Robertson effect, and the damped oscillator. The presentation of the Hamilton-Jacobi theory has been made more concise and a chapter on special applications of this theory has been added containing sections on non-central forces, spin motion, variational principles in rocket motion, the Boltzmann and Navier-Stokes equations. A chapter on continuous media and fields has been added and the chapter on orbits of particles in high energy accelerators has been considerably expanded. The new edition has four appendices dealing with Riemannian geometry, linear vector spaces, group theory and molecular vibration, and quaternions and Pauli matrices; the second of these replaces a chapter of the earlier edition.

W. PRAGER

Theoretical hydrodynamics. By L. M. Milne-Thomson. Fourth edition. The Macmillan Co., New York, 1960. xxviii + 660 pp. $11.00.

According to a statement in the preface, this edition differs from the preceding one primarily by the addition of articles concerned with the following subject: the formulas of Plemilj, flows under gravity with a free surface, an exact treatment of the surface wave of constant form and the resulting "exact linearized theory," and comparison theorems including Serrin's "under-over" theorem.

Theory of thermal stresses. By B. A. Boley and J. H. Weiner. John Wiley & Sons, Inc., New York, 1960. xv + 586 pp. $15.50.

Interest in thermal stresses and their importance in engineering science have increased rapidly during the last decade. The book presents the extensive material in a clear manner. Written in a broad style, it is divided into two main parts. In the first part, the mechanical and thermodynamical foundations of the theory are laid. Although the laws of thermodynamics are introduced in a very general form, their application to the thermal stress problem is restricted from the outset to small deformations and to linear elastic and viscoelastic media. A number of uniqueness proofs for various cases is given in considerable detail. The—in general negligible—influence of inertia effects and of the coupling between the heat equation and the stress-strain equations is thoroughly and clearly discussed. Illustrative examples are given. Of course, most of the particular problems treated are of the uncoupled, quasi-static type.

A long chapter—more than one hundred pages—is devoted to a presentation of the theory of heat conduction. In the opinion of the reviewer this inclusion makes the book unnecessarily bulky. The excellent treatise by Carslaw and Jaeger will have to be consulted anyway by everyone interested in thermal stress and, hence, in heat conduction.

The second part of the book is, apart from a few cross-references, completely independent of the first. The basic equations are developed once more; this time, however, from a more intuitive point of view. First, only purely elastic systems are considered, coupling effects and, with two exceptions, inertia effects being neglected. Beams, built-up structures and plates are treated in detail. A chapter on thermoelastic stability is included. The final three chapters deal with inelastic systems and present a very good introduction to linear and nonlinear viscoelastic and to plastic stress analysis.

It seems that the book, which to a considerable extent is based on original work of the authors and their coworkers, has been written with two different types of readers in mind: the scientist who wants

(*Continued on p. 173*)

— NOTES —

A NEW INTERPRETATION OF THE PLASTIC MINIMUM PRINCIPLES*

BY PHILIP G. HODGE, JR. (*Illinois Institute of Technology, Chicago, Illinois*)

In the theory of elasticity, the principle of minimum potential energy states that

$$\Pi \equiv \int_V \left(\int \sigma_{ij}\, d\epsilon_{ij} \right) dV - \int_{S_T} T_i u_i\, dS - \int_V F_i u_i\, dV \tag{1}$$

is a minimum for the actual state among all kinematically admissible states. A mathematically similar principle for elastic/plastic materials states that

$$W(\sigma_*' , \epsilon_*') - W(\sigma', \epsilon') \geq 0, \tag{2}$$

where the functional W is defined for any independent states of stress σ_{ij}^0 and strain ϵ_{ij}^* by

$$W(\sigma_0 , \epsilon_*) = \int_V \sigma_{ij}^0 \epsilon_{ij}^*\, dV - \int_{S_T} T_i^0 u_i^*\, dS - \int_V F_i^0 u_i^*\, dV \tag{3}$$

and primes denote differentiation with respect to t. We shall consistently use the asterisk to denote an arbitrary kinematically admissible state and unmarked symbols to denote the actual state. Precise definitions of terms, proofs of (2), and credits for original references may be found in any of several books such as [1, 2, 3, 4, 5].[1]

Despite the familiarity of (2), the principle has never received a name. This is doubtless due to the fact that (2) has the unwieldy dimensions of rate of power and it is difficult to find any physical significance for $W(\sigma', \epsilon')$. The purpose of the present note is to show that the principle embodied in (2) may be given either of the following two interpretations.

Theorem. Among all kinematically admissible rate states the actual rate state minimizes the time variations of

(*1*) *the potential dissipation function* $W(\sigma, \epsilon')$;

(*2*) *the pseudo-potential*[2] *energy* Π.

The proofs of these two assertions rest on the principle of virtual work and related principles involving rates. In the present notation

$$W(\sigma, \epsilon_*^{(t)}) = W(\sigma, \epsilon^{(t)}), \tag{4}$$

where the superscript (t) stands for any number of time differentiations. Also, we note that since the existing state is always known and only the rates are varied, only rate quantities need be distinguished by an asterisk. Therefore, we can write

$$\Pi_*' - \Pi' = W(\sigma, \epsilon_*') - W(\sigma, \epsilon') = 0 \tag{5}$$

and

$$\Pi_*'' - \Pi'' = W(\sigma, \epsilon_*'') - W(\sigma, \epsilon'') + W(\sigma_*', \epsilon_*') - W(\sigma', \epsilon') \geq 0. \tag{6}$$

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[1] Numbers in brackets refer to the list of references collected at the end of the paper.

[2] Since $\int \sigma_{ij} d\epsilon_{ij}$ may be partly dissipated for a plastic material, Π as defined by (1) is no longer a potential energy.

Equation (5) is an immediate consequence of (4), and the inequality (6) follows from (5) and (2). The first statement of the theorem is thus verified and the second statement becomes evident when $\Pi(t + \Delta t)$ is expanded in a Taylor's series in Δt about the time t.

If the complementary energy is defined by

$$\Pi_c \equiv \int_V \left(\int \epsilon_{ij} \, d\sigma_{ij} \right) dV - \int_{S_D} T_i u_i \tag{7}$$

and a functional W_c by

$$W_c(\sigma_0 \, , \epsilon_*) = \int_V \sigma^0_{ij} \epsilon^*_{ij} - \int_{S_D} T^0_i u^*_i \tag{8}$$

then the elastic principle of complementary energy states that Π_c is a minimum for the actual state among all statically admissible states and the analogous plastic principle states that

$$W_c(\sigma'_0 \, , \epsilon'_0) - W_c(\sigma', \epsilon') \geq 0. \tag{9}$$

Just as for the first principle one can easily prove two consequences of (9).

Theorem. Among all statically admissible rate states the actual rate state minimizes the time variations of

(1) the complementary dissipation function $W_c(\sigma', \epsilon)$:
(2) the complementary energy Π_c .

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ON THE SIMULTANEOUS DIAGONALIZATION OF TWO SEMI-DEFINITE MATRICES*

BY ROBERT W. NEWCOMB (*University of California, Berkeley*)

1. Introduction. The use of congruency transformations for simultaneously diagonalizing two symmetric matrices, one of which is definite, is well known. One merely diagonalizes the definite matrix to (plus or minus) unity. This is then followed by an orthogonal transformation which diagonalizes the other matrix while preserving the unit matrix already obtained [1]. If, instead of being definite, one matrix is semi-definite,

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this method fails. However, if both matrices are semi-definite, this standard procedure can be extended.

2. Diagonalization. In the following let the superscript t denote matrix transposition and 1_r denote the unit matrix of order r. Further let the rth order zero matrix be denoted by 0_r. The main result is then the following theorem.

Theorem: Let A and B be $n \times n$ real, symmetric, positive semi-definite matrices. Then there exists a real non-singular matrix T and real diagonal matrices A_0 and B_0, [see Eqs. (3) & (8)], such that

$$\begin{aligned} A &= T^t A_0 T \\ B &= T^t B_0 T \end{aligned} \tag{1}$$

Proof: Let A have rank a and B rank b and assume that $b \geq a$. We first find a real, non-singular T_0 such that

$$\begin{aligned} A &= T_0^t A_0 T_0 \\ B &= T^t{}_0 B' T_0 \end{aligned} \tag{2}$$

where

$$A_0 = \text{diagonal}\,[1_a\,,\ 0_{n-a}]. \tag{3}$$

If any of the last $n - a$ diagonal elements of B' are zero, the corresponding entire row and column of B' are zero, since B' is semi-definite. For the last $n - a$ diagonal elements of B' which are nonzero, we can reduce the remaining nondiagonal elements in these rows and columns to zero. We must do this by always adding the diagonal element to the off diagonal element in order to preserve A_0. We can then write

$$\begin{aligned} A &= T_0^t T_1^t A_0 T_1 T_0 \\ B &= T_0^t T_1^t B'' T_1 T_0 \end{aligned} \tag{4}$$

where

$$B'' = \begin{bmatrix} B_a & & 0 \\ & 1_{b-\beta} & \\ 0 & & 0_{n-a-b+\beta} \end{bmatrix} \tag{5}$$

Here $\beta \geq 0$ is defined as the rank of B_a. We now diagonalize B_a by an orthogonal transformation T_a and put

$$T_2 = \begin{bmatrix} T_a & 0 \\ 0 & 1_{n-a} \end{bmatrix} \tag{6}$$

Now let

$$T = T_2 T_1 T_0. \tag{7}$$

Then Eq. (1) results with

$$B_0 = \text{diagonal}\,[\lambda_1\,,\ \cdots\,,\ \lambda_\beta\,,\ 0_{a-\beta}\,,\ 1_{b-\beta}\,,\ 0_{n-a-b+\beta}] \tag{8}$$

where $\lambda_i > 0$, $i = 1, \cdots, \beta$.

By observing that neither the "sign" of A_0 nor that of B_0 enters into the proof, we see that we can diagonalize two semi-definite matrices (possibly of opposite sign). We can also easily extend the theorem to Hermitian matrices. Thus let a superscript asterisk denote complex conjugation and let A and B be complex Hermitian, positive semi-definite matrices. Proceeding as above, but using complex T_0 , T_1 and unitary T_a , we can write

$$\begin{aligned} A &= T'^{*} A_a T \\ B &= T'^{*} B_a T \end{aligned} \tag{9}$$

where A_0 and B_0 are as in Eqs. (3) and (8).

3. Applications. The above theorem is necessary for the synthesis of networks which are passive or active at a point (to be published, for the basic concepts see [2]). It also can be used to advantage in the synthesis of two element kind networks, as well as in studying equivalent networks (see pp. 96 and 142 of [3]). Its use in studying the vibrations of systems satisfying Lagrange's equations should also be apparent.

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AN UPPER BOUND ON RIGHT HALF PLANE ZEROS*

By DOV HAZONY (*Case Institute of Technology*)

Abstract. An upper bound is placed on the number of right half plane zeros of functions of the type $Z - m/n$. Z and m/n are RLC and LC driving point impedance functions respectively. In addition, it is shown that if $\mathrm{Re}Z > 0$ on j axis, the number of right half plane zeros is determined precisely.

Introduction. In problems of control and network synthesis, it may be necessary to determine the number of right half plane zeros of certain impedance functions. In control problems, zeros in the right half plane may cause instability while in synthesis they may require active networks. In this paper an upper bound is placed on the number of these zeros of the class of functions $Z - m/n$ and $Z - n/m$. These terms are defined below.

Lemma.

Given: I. Z is prf (an RLC driving point impedance function).
II. $m + n$ is a Hurwitz polynomial, of degree d, of the complex variable S; m is an even and n is an odd function of S.

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IIIa. The real part of Z, ReZ, may have some isolated zeros on the j axis but it cannot be identically zero along any segment of the j axis. (Part 1).

or IIIb. Re$Z > 0$ on j axis. (Part 2).

Then $mz-n$ and $m-nz$ have, at most, d zeros each in the right half S plane(Part 1) and exactly d zeros if IIIb is exposed (Part 2).

Part 1. *Proof.* Let ζ be given by

$$\zeta = (mZ - n)/(m - nZ) \tag{1}$$

from which follows

$$\left(\frac{Z-1}{Z+1}\right) = \left(\frac{m-n}{m+n}\right)\left(\frac{\zeta-1}{\zeta+1}\right). \tag{2}$$

Since Z is prf (by I) and $|(m-n)/(m+n)| = 1$ on the j axis (by II)

$$\left|\frac{Z-1}{Z+1}\right| = \left|\frac{\zeta-1}{\zeta+1}\right| \leq 1 \quad \text{on the } j \text{ axis.} \tag{3}$$

Let

$$w = \frac{(Z-1)/(Z+1)}{(m-n)/(m+n)} = \frac{\zeta-1}{\zeta+1} \quad \text{and} \quad \therefore \quad |w| \leq 1 \quad \text{on } j \text{ axis.} \tag{4}$$

To prove the lemma, it will suffice to show that ζ has as many pole zero pairs in the right half plane as w has poles, since w may not have more than d poles in the right half plane. From (4) the following can be verified

$$\begin{aligned} &\text{Re}\,\zeta > 0 \quad \text{for} \quad |w| < 1, \\ &\text{Re}\,\zeta = 0 \quad \text{for} \quad |w| = 1, \\ &\text{Re}\,\zeta < 0 \quad \text{for} \quad |w| > 1. \end{aligned} \tag{5}$$

Let D be any S plane domain such that (Fig. 1)

$$\begin{aligned} &|w| = 1 \quad \text{along the boundaries of } D, \\ &|w| > 1 \quad \text{inside } D. \end{aligned} \tag{6}$$

Then, by the maximum modulus theorem, w has at least one pole inside D. Let ζ be given by $|\zeta|e^{j\alpha}$. Then by (5) and (6)

$$\begin{aligned} &|\alpha| = 90^\circ \text{ along the boundary of } D, \\ &|\alpha| > 90^\circ \text{ inside } D, \\ &|\alpha| < 90^\circ \text{ outside } D. \end{aligned} \tag{7}$$

Now, since the 90° phase locus must start at a zero and end at a pole, it follows that all poles and zeros of ζ lie on the boundary of D and occur in pairs. Further, any phase locus starts at a zero and ends at a pole. Then, inside D, ζ has as many 180° phase loci as the number of pole zero pairs it has on the boundary of D. Along each 180° phase loci the magnitude of ζ increases monotonically* from 0 to ∞. Then, at one point along this

*This is due to the orthogonal properties of magnitude and phase plots.

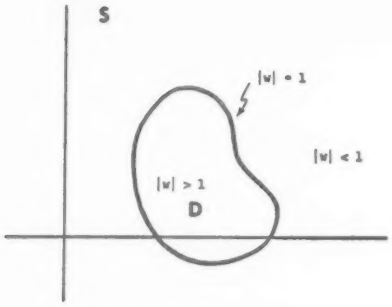

FIG. 1 A plot of w in the S plane.

line, $\zeta = -1$ and w has a pole. It follows that ζ has as many pole zero pairs on the boundary of D as the number of poles w has inside. In addition, the following properties of w in the *right half S plane* may be ascertained:

w has d poles at most. (by (4))
If w has poles it also has at least one domain D.
This D must lie exclusively in the right half plane.
(by postulate III and because $| w | \leq 1$ on the j axis, by (4)) (8)

It follows that ζ has d poles and d zeros in the right half S plane at most. This completes the proof (Part 1).

Part 2. Rouche's theorem [1] will be used to prove Part 2.

Rouche's theorem. If $P(s)$ and $Q(s)$ are analytic interior to a simple closed Jordan curve C and if they are continuous on C and

$$| P(s) | < | Q(s) | \quad \text{on} \quad C \tag{9}$$

then the function $F(s) = P(s) + Q(s)$ has the same number of zeros interior to C as does $Q(s)$.

Let $Q(s) = \frac{1}{2}(n - m)/(m + n)$, $Q(s)$ has d right half plane zeros and $| Q(s) | = \frac{1}{2}$ on j axis (by II).
and $P(s) = \frac{1}{2}(Z - 1)/(Z + 1)$. $| P(s) | < \frac{1}{2}$ on j axis as $\mathrm{Re}Z > 0$ there
from which it follows that $F(s) = (nZ - m)/(m + n)(Z + 1)$ and therefore also $nZ - m$ have d right half plane zeros. Similarly, by letting $Q(s) = \frac{1}{2}(m - n)/(m + n)$, $mZ - n$ has d right half plane zeros.

This completes the proof.

Applications. *Example 1.* In the following network (Fig. 2) $Z = \mathrm{prf}$, $m/n = a$ reactance function ($m + n$ is of degree d). Then, the transfer function,

$$Z_{21} = \frac{E_2}{E_1}\bigg|_{I_2=0} = \frac{nZ - m}{nZ + m} \tag{10}$$

has, at most, d right half plane zeros.

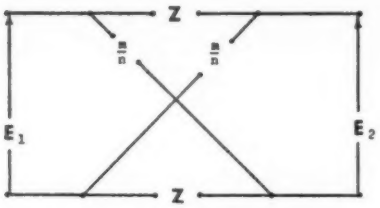

FIG. 2 A lattice in which $\left.\frac{E_2}{E_1}\right|_{I_2=0} = \frac{nZ - m}{nZ + m}$.

Example 2. In the following network [2] (Fig. 3) ζ is given by (Z is prf)

$$\zeta = \left[\left(1 + \frac{S^2}{ab}\frac{bZ(a) - aZ(b)}{bZ(b) - aZ(a)}\right)Z - S\frac{Z(a)Z(b)}{ab}\frac{b^2 - a^2}{bZ(b) - aZ(a)}\right] \cdot \left[\left(1 + \frac{S^2}{ab}\frac{bZ(b) - aZ(a)}{bZ(a) - aZ(b)}\right) - S\frac{Z}{ab}\frac{b^2 - a^2}{bZ(a) - aZ(b)}\right]^{-1}$$

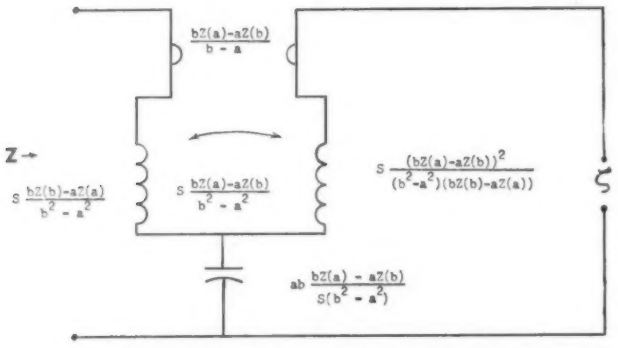

FIG. 3 Physical realization of an impedance function, Z, in terms of ζ.

where a and b are positive real or complex conjugates with a positive real part. ζ is constructed to have the factor $(s - a)(s - b)$ common in both the numerator and denominator. Further, [2] Re $\zeta \geq 0$ on the j axis. To complete the proof of the realizability of ζ it is necessary to show that no additional right half plane zeros and poles are present. By the above lemma, both numerator and denominator have only two right half plane zeros each. This completes the proof.

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THE DISTRIBUTION OF QUADRATIC FUNCTIONALS*

BY JOHN B. THOMAS (*Princeton University*) AND EUGENE WONG (*IBM Research Center*)

1. Introduction. The purpose of this note is to consider the joint probability distribution of quadratic functionals of the form

$$y_m = \int_0^t \sum_{i,k} x_i(\tau) h_{ik}^{(m)}(\tau) x_k(\tau)\, d\tau, \tag{1}$$

where the $x_i(\tau)$ are correlated Gaussian processes and the $h_{ik}^{(m)}(\tau)$ are given functions. The probability distribution of a single functional of the form of (1) has been treated by another method Siegert [1] for the case where the $x_i(\tau)$ are stationary.

The solution to the problem described reduces to the solution of a homogeneous matrix integral equation. This equation shows explicitly the transformation functions $h_{ik}^{(m)}$ and the covariance functions of the $x_i(\tau)$.

2. Analysis. Let the functionals under consideration be denoted by

$$y_m = \int_0^t X(\tau) \cdot h^{(m)}(\tau) X(\tau)\, d\tau, \qquad m = 1, 2, \cdots, p. \tag{2}$$

Here $X(\tau)$ is an n-element column vector whose elements x_i are correlated Gaussian processes with zero means, and $h^{(m)}(\tau)$ is an $n \times n$ symmetric matrix [2] whose elements $h_{ik}^{(m)}$ are given functions. In addition let $X(\tau)$ be written formally as the series

$$X(\tau) = \sum_{k=1}^{\infty} a_k \phi_k(\tau), \tag{3}$$

where the a_k are random variables and the $\phi_k(\tau)$ are non-random vector functions. With the substitution of this series, (2) becomes

$$y_m = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j a_k \int_0^t \phi_j(\tau) \cdot h^{(m)}(\tau) \phi_k(\tau)\, d\tau. \tag{4}$$

The characteristic function for the joint probability distribution of the y_m is given by

$$F(\eta_1, \cdots, \eta_p) \triangleq E\left\{\exp \sum_{m=1}^{p} i\eta_m y_m\right\}, \tag{5}$$

where $i = (-1)^{1/2}$ and $E\{\ \}$ denotes the expectation of the bracketed quantity. From (4), the characteristic function becomes

$$F(\eta_1, \cdots, \eta_p) = E\left\{\exp i \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j a_k \int_0^t \phi_j(\tau) \cdot w(\tau) \phi_k(\tau)\, d\tau\right\}, \tag{6}$$

where the order of summation has been interchanged and the weighting function matrix $w(\tau)$ has been defined by

$$w(\tau) \triangleq \sum_{j=1}^{p} \eta_j h^{(j)}(\tau). \tag{7}$$

Let the elements of the covariance function matrix of the $x_i(\tau)$ be given by

$$R_{ik}(\tau, \sigma) \triangleq E\{x_i(\tau) x_k(\sigma)\}, \tag{8}$$

*Received Aug. 12, 1960.

and let the $\phi_m(\tau)$ be the eigenfunctions of the homogeneous matrix integral equation

$$\int_0^t R(\tau, \sigma)w(\sigma)\phi(\sigma)\, d\sigma = \lambda\phi(\tau). \tag{9}$$

Then it is shown in the next section that the $\phi_m(\tau)$ are orthonormal with respect to the weighting function $w(\tau)$, i.e.,

$$\int_0^t \phi_k(\tau)\cdot w(\tau)\phi_l(\tau)\, d\tau = \delta_{kl}\ , \tag{10}$$

and the a_k are uncorrelated; hence the a_k are independent Gaussian variables with zero means and variances λ_k . The characteristic function can be written from (6) and (9) as

$$F(\eta_1\ , \cdots\ , \eta_p) = E\left\{\exp i \sum_{k=1}^{\infty} a_k^2\right\} = \prod_{k=1}^{\infty} (1 - 2i\lambda_k)^{-1/2}. \tag{11}$$

Equation (11) is the solution to the problem provided that the series representation (3) is valid and that the eigenvalues λ_k can be found from (9). These points will now be discussed.

3. Series representation. It will be shown that the vector eigenfunctions of (9) which correspond to different eigenvalues are orthogonal with respect to the weighting function matrix $w(\tau)$. This proof parallels the standard proof for the scalar case [3].

For two distinct eigenvalues λ_k and λ_j , expression (9) can be used to form

$$\int_0^t \phi_k(\tau)\cdot w(\tau)\phi_j(\tau)\, d\tau = \frac{1}{\lambda_k}\int_0^t \int_0^t R(\tau, \sigma)w(\sigma)\phi_k(\sigma)\cdot w(\tau)\phi_j(\tau)\, d\sigma\, d\tau, \tag{12}$$

and

$$\int_0^t \phi_k(\tau)\cdot w(\tau)\phi_j(\tau)\, d\tau = \frac{1}{\lambda_j}\int_0^t \int_0^t \phi_k(\tau)\cdot w(\tau)R(\tau, \sigma)w(\sigma)\phi_j(\sigma)\, d\tau\, d\sigma. \tag{13}$$

The matrix identity

$$MA\cdot B = A\cdot M_T B,$$

where the subscript T is used to denote the transpose of the $n \times n$ matrix M and where A and B are n-element column vectors, may be applied to the right side of (12) to yield

$$\int_0^t \phi_k(\tau)\cdot w(\tau)\phi_j(\tau)\, d\tau = \frac{1}{\lambda_k}\int_0^t \int_0^t \phi_k(\sigma)\cdot w_T(\sigma)R_T(\tau, \sigma)w(\tau)\phi_j(\tau)\, d\sigma\, d\tau. \tag{14}$$

Now, if σ and τ are interchanged in (14) and if it is noted that $w_T(\tau) = w(\tau)$ and that $R_T(\sigma, \tau) = R(\tau, \sigma)$, the result is

$$\int_0^t \phi_k(\tau)\cdot w(\tau)\phi_j(\tau)\, d\tau = \frac{1}{\lambda_k}\int_0^t \int_0^t \phi_k(\tau)\cdot w(\tau)R(\tau, \sigma)w(\sigma)\phi_j(\sigma)\, d\tau\, d\sigma. \tag{15}$$

A comparison of (13) and (15) shows that, for $\lambda_j \neq \lambda_k$,

$$\int_0^t \phi_k(\tau)\cdot w(\tau)\phi_j(\tau)\, d\tau = 0 \tag{16}$$

and, with proper normalization, (10) follows.

That the a_k of (3) are uncorrelated as a consequence of (9) may be shown by forming $E\{a_k a_i\}$. From (10),

$$E\{a_k a_i\} = E\left\{\int_0^t \int_0^t [X(\tau)\cdot w(\tau)\phi_k(\tau)][X(\sigma)\cdot w(\sigma)\phi_i(\sigma)]\, d\tau\, d\sigma\right\}. \tag{17}$$

The integrand of (17) may be rearranged to yield

$$E\{a_k a_i\} = \int_0^t \left[\int_0^t E\{X(\tau)X_T(\sigma)\}w(\sigma)\phi_i(\sigma)\, d\sigma\right]\cdot w(\tau)\phi_k(\tau)\, d\tau. \tag{18}$$

Since $E\{X(\tau)X_T(\sigma)\} = R(\tau, \sigma)$, it follows from (9) that

$$E\{a_k a_i\} = \lambda_i \int_0^t \phi_i(\tau)\cdot w(\tau)\phi_k(\tau)\, d\tau, \tag{19}$$

and, from (10), that

$$E\{a_k a_i\} = \lambda_i\, \delta_{ik}\,. \tag{20}$$

Thus the a_k are uncorrelated random variables with variances λ_k . The means of the a_k are each zero, i.e.

$$E\{a_k\} = \int_0^t E\{X(\tau)\}\cdot w(\tau)\phi_k(\tau)\, d\tau = 0, \tag{21}$$

and the linearity of (10) makes the a_k Gaussian.

The series expansion (3) is a generalization of the Karhunen-Loeve representation [4, 5] and reduces to that representation for the case of one-dimension and for $w(x) \equiv 1$. As in the one-dimensional case, it can be shown [6, 7] that the ϕ_i exist and that the components of the vector series (3) converge mean square to the components of the vector process $X(\tau)$. In addition, if $R(\tau, \sigma)$ is a positive definite kernel, then the ϕ_i are complete and the λ_i have at most a finite degeneracy. In these proofs a generalization of Mercer's theorem is required to represent the elements of the matrix $R(\tau, \sigma)$.

4. The single-functional case. For the special case where only a single functional y is considered, (9) can be written as

$$\int_0^t R(\tau, \sigma)h(\sigma)\phi(\sigma)\, d\sigma = \nu\phi(\tau), \tag{22}$$

and the characteristic function (11) becomes

$$F(\eta) = \prod_{k=1}^{\infty} (1 - 2i\eta\nu_k)^{-1/2}, \tag{23}$$

where $\eta\nu_k = \lambda_k$. Now if the matrix $g(\tau, \mu; \eta)$ is defined by

$$g_{lm}(\tau, \mu; \eta) \triangleq \sum_{k=1}^{\infty} \frac{\nu_k \phi_k^{(l)}(\tau)\phi_k^{(m)}(\mu)}{1 - 2i\eta\nu_k}\,, \tag{24}$$

then, by direct evaluation,

$$g(\tau, \mu; \eta) - 2i\eta \int_0^t R(\tau, \sigma)h(\sigma)g(\sigma, \mu; \eta)\, d\sigma = R(\tau, \mu). \tag{25}$$

For the stationary case (25) is identical, except for a change in notation, with Eq. (79) of Ref. [1]. In terms of the matrix $g(\tau, \mu, \eta)$ the characteristic function can be written

$$F(\eta) = \exp i \int_0^{\eta} d\xi \int_0^{t} Tr[g(\sigma, \sigma; \xi)h(\sigma)]\, d\sigma, \tag{26}$$

where $Tr[\ ]$ is the trace of the matrix.

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FURTHER EXTENSIONS OF SCHUSTER'S INTEGRAL*

By E. T. KORNHAUSER** (*H. H. Wills Physics Laboratory, University of Bristol*)

The integral,

$$I = \int_0^{\infty} [C^2(x) + S^2(x)]\, dx,$$

where $C(x)$ and $S(x)$ are Fresnel integrals defined by

$$C(x) = \int_x^{\infty} \cos t^2\, dt,$$

$$S(x) = \int_x^{\infty} \sin t^2\, dt,$$

was conjectured by Schuster[1] to have the value $(\pi/8)^{\frac{1}{2}}$. Proof that I does in fact have this value was given by Hardy[2] and more elegantly by Ingham[3]. More recently Bateman[4] has extended Ingham's treatment to evaluate integrals of the form

$$\int_0^{\infty} C(x)C(ax)\, dx, \qquad \int_0^{\infty} C(x)S(ax)\, dx,$$

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**On leave from Brown University.

[1]Sir A. Schuster, Proc. Roy. Soc. **107**, 15 (1925).

[2]G. H. Hardy, Proc. London Math. Soc. **24**, xxx (1926)

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[4]H. Bateman, Proc. Natl. Acad. Sci. **32**, 70 (1946).

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etc. An additional generalization, which is of interest in the theory of imperfect diffraction gratings[5], involves the inclusion of additive rather than multiplicative constants in the argument of the Fresnel functions.

Consider the integral

$$I_1 = \int_0^\infty C^2(x + a)\,dx = \int_a^\infty C^2(x)\,dx = [xC^2(x)\,|_a^\infty + 2\int_a^\infty x\cos x^2 C(x)\,dx$$

$$= -aC^2(a) + [\sin x^2 C(x)\,|_a^\infty + \int_a^\infty \sin x^2 \cos x^2\,dx$$

$$= -aC^2(a) - \sin a^2 C(a) + \tfrac{1}{2}\,2^{-1/2}S(2^{1/2}a).$$

In like manner

$$I_2 = \int_0^\infty S^2(x + a)\,dx = -aS^2(a) + \cos a^2 S(a) - \tfrac{1}{2}\,2^{-1/2}S(2^{1/2}a).$$

Furthermore,

$$I_3 = \int_0^\infty [C^2(x + a) + S^2(x + a)]\,dx = I_1 + I_2$$

$$= -a[C^2(a) + S^2(a)] + \cos a^2 S(a) - \sin a^2 C(a).$$

Changing the sign of a yields

$$I_4 = \int_0^\infty [C^2(x - a) + S^2(x - a)]\,dx$$

$$= a[C^2(-a) + S^2(-a)] + \cos a^2 S(-a) - \sin a^2 C(-a),$$

and since $C(-a) = (\pi/2)^{\frac{1}{2}} - C(a)$ and similarly for $S(-a)$,

$$I_3 + I_4 = (\pi/2)^{1/2}(\cos a^2 - \sin a^2) + 2a(\pi/2)^{1/2}[(\pi/2)^{1/2} - C(a) - S(a)].$$

Now consider the integral,

$$I_5 = \int_0^\infty [C(x + a)C(x - a) + S(x + a)S(x - a)]\,dx.$$

Successive integrations by parts yield

$$I_5 = \tfrac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) + a\int_0^\infty [C(x + a)\cos(x - a)^2$$

$$- C(x - a)\cos(x + a)^2 + S(x + a)\sin(x - a)^2 - S(x - a)\sin(x + a)^2]\,dx,$$

which may be written

$$I_5 = \tfrac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) + a\int_0^\infty \frac{\partial}{\partial a}[C(x + a)C(x - a) + S(x + a)S(x - a)]\,dx$$

Thus it is clear that I_5 obeys the differential equation,

$$I_5 - a\frac{dI_5}{da} = \tfrac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) \equiv F(a)$$

which may be solved by quadrature,

$$I_5 = a\int_a^\infty u^{-2}F(u)\,du = \tfrac{1}{2}(\pi/2)^{1/2}\{\cos a^2 - \sin a^2 - 2a[C(a) + S(a)]\}.$$

Finally, the most significant integral from the point of view of the physical problem

$$I_6 = \int_0^\infty \{[C(x+a) - C(x-a)]^2 + [S(x+a) - S(x-a)]^2\}\,dx$$

$$= I_3 + I_4 - 2I_5 = \pi a,$$

a remarkably simple result.

NOTE ON THE POINCARÉ BOUNDARY-VALUE PROBLEM*

By E. E. JONES (*University of Nottingham, England*)

1. This note is concerned with the solution of a modified form of the Poincaré boundary-value problem [1]. It is required to solve the Poisson differential equation $\nabla^2\phi_i = f(x, y)$ for $\phi_i(x, y)$ defined in S_i, the region enclosed by the circle C of equation $|z| = a$, $(z = x + iy)$, such that on C

$$k\frac{\partial\phi_i}{\partial n} + l\frac{\partial\phi_i}{\partial s} + m\phi_i = g(x, y), \tag{1}$$

where k, l, m are constants, $f(x, y)$ is prescribed in S_i, $g(x, y)$ is prescribed on C, and $\partial/\partial n$, $\partial/\partial s$ denote differentiations along the inward normal and positive tangential directions respectively to C.

It is assumed that $\phi_i = \phi_{i0} + \Phi_i$, where $\phi_{i0}(x, y)$ is a particular integral of the Poisson equation reflecting all the singularities of the complete solution ϕ_i, and $\Phi_i(x, y)$ is harmonic in S_i, being together with its first partial derivatives single-valued and continuous in S_i. It is thus possible to write $\Phi_i = re\,W_i(z)$, where $W_i(z)$ is a regular function of z in S_i. If $z = ae^{i\theta} = \zeta$, $\bar{z} = a^2/\zeta$ on C, and by definition

$$h(\zeta) = -g - k\frac{\partial\phi_{i0}}{\partial r} + \frac{l}{a}\frac{\partial\phi_{i0}}{\partial\theta} + m\phi_{i0}, \tag{2}$$

then

$$h_i(z) = \frac{1}{2\pi i}\int_C \frac{(\zeta + z)h(\zeta)}{\zeta - z}\,d\zeta, \tag{3}$$

is the Schwarz integral representation of a function $h_i(z)$ regular in S_i with a real part equal to $h(\zeta)$ on C—for this to be so it is necessary for $h(\zeta)$ to satisfy the Lipschitz condition on C [2]. The boundary condition (1) then takes the form

$$re\left\{\frac{\zeta}{\alpha}\frac{dW_i(\zeta)}{d\zeta} - mW_i(\zeta) - h_i(\zeta)\right\} = 0, \tag{4}$$

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where $\alpha = a/(k - il)$. When $re(m\alpha) < 0$, then a unique solution exists for $W_i(z)$ in S_i in the form

$$W_i(z) = \alpha \int_0^1 h_i(zt)t^{-m\alpha-1}\,dt. \tag{5}$$

This form of expression has an advantage over that given in [1], since it also includes the conditions for the existence of a solution. Thus when $m = 0$ the equation (5) determines $W_i(z)$ to within an arbitrary constant, provided the integral is uniformly convergent, which necessitates $h_i(0) = 0$. It thus follows that since the integral of (5) is uniformly convergent for $re(m\alpha) < 0$, and for $m = 0$ if $h_i(0) = 0$, then $W_i(z)$ is a regular function of z in S_i , as required, and $W_i'(z)$ can be obtained by differentiation under the integral sign [3].

Repeated integration by parts extends the range to all positive values of $re(m\alpha)$, including $m\alpha$ a positive integer—in this latter case solutions exist but are not uniquely determined [1].

2. The particular integral ϕ_{i0} can be determined by integrating the Poisson differential equation in the form

$$4\frac{\partial^2\phi_{i0}}{\partial z\,\partial\bar{z}} = f[x(z,\bar{z}),\, y(z,\bar{z})],$$

and the solution can be written as

$$\phi_{i0} = \phi_{i0}^* + re\Omega_i(z),$$

where $\Omega_i(z)$ is a function of z, regular outside C, having prescribed singularities in S_i only. It follows that (2) can be written as

$$h(\zeta) = h^*(\zeta) + re\left\{-\frac{a^2}{\bar{\alpha}\zeta}\,\bar{\Omega}_i'\left(\frac{a^2}{\zeta}\right) + m\bar{\Omega}_i\left(\frac{a^2}{\zeta}\right)\right\}.$$

But $h(\zeta) = \{re\, h_i(z)\}_C$, then evidently

$$h_i(z) = h_i^*(z) - \frac{a^2}{\bar{\alpha}z}\,\bar{\Omega}_i'\left(\frac{a^2}{z}\right) + m\bar{\Omega}_i\left(\frac{a^2}{z}\right),$$

which is a regular function of z in S_i , as required. On substitution into (5), and integrating by parts, then readily

$$W_i(z) = W_i^*(z) + \frac{\alpha}{\bar{\alpha}}\,\bar{\Omega}_i\left(\frac{a^2}{z}\right) + \frac{2mk\alpha^2}{a}\int_1^\infty \bar{\Omega}_i\left(\frac{a^2t}{z}\right)t^{m\alpha-1}\,dt. \tag{6}$$

This result determines $W_i(z)$ when the singularities of $\Omega_i(z)$ are prescribed, and extends the results of Ludford et al. [4].

A similar type of reasoning leads to the conclusion that it is possible to add the real part of any function of z regular in S_i to ϕ_{i0} without changing the value of ϕ_i .

3. For the external problem with ϕ_e defined in S_e , the region $|z| > a$, and satisfying a boundary condition similar to (1) with $\partial/\partial n$ denoting differentiation along the normal to C drawn into S_e , the solution analogous to (6) is

$$W_e(z) = W_e^*(z) + \frac{\bar{\alpha}}{\alpha}\,\bar{\Omega}_e\left(\frac{a^2}{z}\right) + \frac{2mk\bar{\alpha}^2}{a}\int_0^1 \Omega_e\left(\frac{a^2t}{z}\right)t^{-m\bar{\alpha}-1}\,dt. \tag{7}$$

Here

$$W_e^*(z) = -\bar{\alpha}\int_1^\infty h_e^*(zt)t^{m\bar{\alpha}-1}\,dt,$$

where $h_e^*(z)$ is regular in S_e , including infinity, such that on C

$$re\ h_e^*(z) = g(x, y) - k\frac{\partial \phi_{e0}^*}{\partial r} - \frac{l}{a}\frac{\partial \phi_{e0}^*}{\partial \theta} - m\phi_{e0}^* ,$$

and $\Omega_e(z)$ is regular in S_i , having singularities only in S_e .

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ON MATRIX DIFFERENTIAL EQUATIONS*

By JOHN JONES, JR., (*Institute of Technology, Air University*)

The purpose of this note is to obtain a necessary condition and a sufficient condition, of an algebraic nature, for the matrix differential equation

$$AX'' + XB = C, \tag{1}$$

where all matrices considered are n-square, to have a single-valued solution. Capital letters denote matrices and prime denotes the derivative with respect to x. The elements of A, B, C belong to the polynomial domain $\mathfrak{F}[x]$ of the field $\mathfrak{F}$ of real numbers. I is the identity matrix.

Theorem 1. If a solution matrix X of Eq. (1) exists, then the following pair of matrices

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \qquad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{2}$$

are equivalent.

Proof. Clearly

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\begin{bmatrix} I & -X'' \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & C & -AX'' & -XB \\ 0 & & B & \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{3}$$

and so the matrices of (2) are equivalent.

Theorem 2. If the matrices of (2) are similar and there exist non-singular constant matrices P, Q such that PAP^{-1}, QBQ^{-1} are diagonal matrices exhibiting the invariant factors a_i , $i = 1, 2, \cdots, \alpha$, and b_j , $j = 1, 2, \cdots, \beta$ of A, B, respectively, along the

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main diagonal, then there exists a solution X of Eq. (1) for $-\infty < a \leq x \leq b < +\infty$ except at the zeros of the invariant factors of A.

Since the matrices of (2) are similar,

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} = \begin{bmatrix} PAP^{-1} & PCQ^{-1} \\ 0 & QBQ^{-1} \end{bmatrix} = M$$
$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} = \begin{bmatrix} PAP^{-1} & 0 \\ 0 & QBQ^{-1} \end{bmatrix} = N, \tag{4}$$

and the matrices M, N are similar. Form the equation

$$DU'' + UF = G, \tag{5}$$

where $D = PAP^{-1}$, $F = QBQ^{-1}$, $G = PCQ^{-1}$. Consider Eq. (5) element-wise, namely,

$$d_{ii}u_{ij}'' + u_{ij}f_{jj} = g_{ij}\,, \qquad (i, j = 1, 2, \cdots, n), \tag{6}$$

where $u_{ij}'' \,\varepsilon\, U''$, $u_{ij} \,\varepsilon\, U$, $d_{ii} \,\varepsilon\, D$, $f_{jj} \,\varepsilon\, F$, and $g_{ij} \,\varepsilon\, G$.

Next to show that Eq. (6) always has a solution. W. E. Roth* has shown that if the matrices of (2) are equivalent, then for the elements g_{ij} of G: (i) g_{ij}, for $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, is a multiple of the greatest common factor of d_{ii} and f_{jj}; (ii) g_{ij} for $1 \leq i \leq \alpha$, $\beta < j \leq n$, is a multiple of d_{ii}; (iii) g_{ij}, for $\alpha < i \leq n$, $1 \leq j \leq \beta$, is a multiple of f_{jj}; (iv) g_{ij}, for $\alpha < i \leq n$, $\beta < j \leq n$, is identically zero. Thus g_{ij} cannot be different from zero when both d_{ii}, f_{jj} are identically zero, and in each of the four cases above, (6) has a solution.

Let $U = P\,X\,Q^{-1}$, then $U'' = PX''Q^{-1} = (PXQ^{-1})''$, so Eq. (5) may be written as

$$(PAP^{-1})(PX''Q^{-1}) + (PXQ^{-1})(QBQ^{-1}) = PCQ^{-1}. \tag{7}$$

Multiplying (8) on the left by P^{-1} and on the right by Q, we have Eq. (1); thus $X = P^{-1}\,UQ$ is a solution of Eq. (1).

FURTHER PROPERTIES OF CERTAIN CLASSES OF TRANSFER FUNCTIONS: II**

By A. H. ZEMANIAN (*College of Engineering, New York University*)

This note is a sequel to a previously published paper [1]. The notation and terminology used here is the same as before. The purpose of this note is to point out a consequence of some previously published results [1, 2], which is immediately applicable to rational transfer functions that have no poles in the right-half plane and have at least twice as many poles as zeros. Such transfer functions arise quite commonly in physical systems.

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*W. E. Roth, *The Equation $AX - YB = C$, and $AX - XB = C$ in Matrices*, Proc. Am. Math. Soc. **3**, 392–396 (1952).

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Let $F(s)$ be such a transfer function. It may be written as follows, where $m \geq 1$.

$$F(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} = \frac{N(s)}{D(s)}. \tag{1}$$

The symbol $f(t)$ will denote the inverse Laplace transform of $F(s)$ that is zero for negative t. (This is the unit impulse response of the system.)

Theorem: Let $F(s)$ be given by (1) *and let $D(s)$ have no roots in the right-half s-plane. For m odd, let $n \leq (m-1)/2$ and, for m even, let $n \leq m/2$. Then*

$$|f(t)| \leq \frac{|a_n|\,t^{m-n-1}}{(m-n-1)!} + \frac{|a_{n-1}|\,t^{m-n}}{(m-n)!} + \cdots + \frac{|a_1|\,t^{m-2}}{(m-2)!} + \frac{|a_0|\,t^{m-1}}{(m-1)!}. \tag{2}$$

Proof. Let $D(s)$ be a Hurwitz polynomial and let $m \geq 2$, at first. By Theorem 3 of [2], $1/D(s)$ is a subclass m function. Hence, by Theorem 2 of [1], $s^\mu/D(s)$ is a class $(m-\mu)$ function where μ is any integer in the range, $0 \leq \mu \leq (m-1)/2$, if m is odd and in the range, $0 \leq \mu \leq m/2$, if m is even. Under these restrictions on μ, the inverse Laplace transform of $s^\mu/D(s)$ is the μ*th* derivative of the inverse Laplace transform of $1/D(s)$ Letting $\mathfrak{L}^{-1}$ denote the inverse Laplace transform, Theorem 3 of [1] implies that

$$\left| \mathfrak{L}^{-1}\left[\frac{s^\mu}{D(s)}\right] \right| \leq \frac{t^{m-n-1}}{(m-n-1)!}.$$

Combining this result with (1), (2) is obtained.

If $m = 1$, the hypothesis implies that $m = 1$ and $n = 0$. In this case, (2) becomes $|f(t)| \leq |a_0|$ and this result is easily established.

The proof is completed by noting that the conclusion still holds when the restriction that $D(s)$ is a Hurwitz polynomial is replaced by the weaker restriction that $D(s)$ has no roots in the right-half s-plane. This is so since the unit impulse response is a continuous function of the pole positions of the corresponding transfer function.

The principal advantage of this theorem over the previously published results is the following. In the previous cases one always had to establish whether a transfer function is a member of some class of functions before a particular bound could be placed on the corresponding unit impulse response. In this case one must establish that the degrees of $N(s)$ and $D(s)$ are appropriate and that $D(s)$ has no poles in the right-half s-plane. The restriction on m and n can be checked by inspection while in many practical cases it is known beforehand that $D(s)$ has no roots with positive, real parts. For instance, if the system under consideration is a lumped, linear, fixed, finite, and passive one, the transfer function will be rational and have no poles in the right-half s-plane.

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ON THE DEFINITION OF STRESS RATE[1]

By E. F. MASUR (*The University of Michigan*)

The concept of stress rate has been the subject of several current papers. Prager [1] has introduced a discussion of some of the proposed definitions from a Cartesian spatial point of view and, in reviewing their suitability for use in the constitutive equations of plasticity, has arrived at a decided preference for Jaumann's [2] definition. Naghdi and Wainwright [3] have generalized the concept of tensor rate in such a way as to include the definitions mentioned by Prager as special cases. Convected coordinates have been employed by Sedov [4] in a discussion of Prager's results. In continuing the approach adopted by Sedov we investigate in the present note the connection between the various stress rates and the concept of a convected derivative introduced by Oldroyd [5]. Additional arguments in favor of Jaumann's definition are established; moreover, a modification of the latter is proposed in connection with the constitutive equations of plasticity.

In order to make this note reasonably self-contained, a brief resumé of Oldroyd's derivation is given in what follows. Let x_i be the instantaneous Cartesian coordinates of a particle [2] and let a^α represent a convected coordinate system defined by $x_i = x_i(a^\alpha, t)$ in such a way that the material coordinates a^α of a particle are time independent. Let $\sigma^{\alpha\beta}$, the contravariant components of stress relative to the material coordinate system in its instantaneous position, be considered functions of the material coordinates a^α and of the time t. The partial derivative of $\sigma^{\alpha\beta}$ with respect to t is called "the convected derivative" by Oldroyd. The components $D_t\sigma^{\alpha\beta}$ so defined form the contravariant components of a tensor, since the functional relationship between the assumed convected coordinate system and some other coordinate system which is also convected is independent of time.

We are therefore justified in defining a tensor σ''' by

$$\sigma'''^{\alpha\beta} = D_t\sigma^{\alpha\beta} \tag{1}$$

and its components relative to the assumed Cartesian spatial system[3] by

$$\sigma'''_{ij} = x_{i,\alpha}x_{j,\beta}\sigma'''^{\alpha\beta} . \tag{2}$$

Since

$$\sigma_{ij} = x_{i,\alpha}x_{j,\beta}\sigma^{\alpha\beta} \tag{3}$$

and

$$D_t x_i = v_i , \tag{4}$$

where v_i represents the Cartesian components of the velocity vector, we obtain by differentiation of Eq. (3) Oldroyd's definition

$$\sigma'''_{ij} = D_t\sigma_{ij} - v_{i,k}\sigma_{jk} - v_{j,k}\sigma_{ik} . \tag{5}$$

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[2]An extension to non-Cartesian spatial systems is straightforward, but contributes nothing essential to the purpose of this discussion.

[3]Greek indices refer to the convected coordinate system and Latin indices to the fixed Cartesian system.

Similarly it is possible to define a tensor σ'' whose covariant components are given by $D_t\sigma_{\alpha\beta}$. Note that the tensor so obtained is in general not the same as the one defined in Eq. (1) (that is, $\sigma''_{\alpha\beta}$ and $g_{\alpha\gamma}g_{\beta\lambda}\sigma'''^{\gamma\lambda}$ are in general different), because the metric tensor $g_{\alpha\beta}$ is time dependent, hence convected time differentiation and lowering and raising of indices are not commutable operations[4]. If the Cartesian components of σ'' are obtained by a process similar to the one employed above, we are led to the definitions of Cotter-Rivlin [8]. Furthermore, if the same process is applied to the Kirchhoff tensor, then the use of the contravariant components leads to Truesdell's [7] definition, while an analogous derivation can be carried out without difficulty in connection with the covariant components.

The four stress rate definitions introduced thus far are rejected by Prager on the basis that their vanishing does not make the stress invariants stationary; consequently their use in the constitutive equations of plasticity leads to complications. This becomes immediately apparent in a convected coordinate system since the use of either contravariant or covariant stress components necessitates the introduction of the time dependent metric tensor in the invariants; the latter are stationary in the presence of stationary stress components only in the case of locally rigid body motion.

This argument for rejection can be strengthened further by the (not unrelated) consideration that the establishment of the constitutive equations of plasticity may make it necessary to separate the stress rate tensor into its isotropic and deviatoric contributions. Moreover, it is reasonable to demand that the vanishing of the stress rates make the deviatoric stress components stationary.

This is not true for any of the definitions introduced thus far. For example, if the contravariant components of the stress deviator, which are defined by

$$s^{\alpha\beta} = \sigma^{\alpha\beta} - \tfrac{1}{3}\sigma^{\gamma\lambda}g_{\gamma\lambda}g^{\alpha\beta} \tag{6}$$

are used for the definition of the stress deviator rate[5]

$$s'''^{\alpha\beta} = D_t s^{\alpha\beta} \tag{7}$$

then it follows that $\sigma''' = 0$ does not necessarily imply $s''' = 0$. Conversely, with the introduction of the strain rate tensor ϵ by means of

$$2\epsilon_{\alpha\beta} = v_{\alpha\,.\beta} + v_{\beta,\,\alpha} = D_t g_{\alpha\beta} \tag{8}$$

the vanishing of s''' implies

$$\sigma'''^{\beta}_{\alpha} = \tfrac{1}{3}\sigma'''^{\gamma}_{\gamma}\,\delta^{\beta}_{\alpha} - \tfrac{2}{3}\sigma^{\gamma}_{\gamma}(\epsilon^{\beta}_{\alpha} - \tfrac{1}{3}\epsilon^{\lambda}_{\lambda}\,\delta^{\beta}_{\alpha}).$$

It is seen that for this case σ''' is not isotropic, as it should be, unless ϵ is isotropic too.

The objections raised against the stress rate definitions introduced above become

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[4]It is equally valid to refer to σ''' and to σ'' (or to any suitably normalized combination thereof) as a "convected derivative" of σ; any rate tensor so defined clearly satisfies the requirement of being insensitive to rigid body motion. The conceptual significance of this ambiguity has apparently not been fully explored by Oldroyd [5], [6] or Truesdell [7], but is implicit in [4].

[5]Note that this defines the rate of the stress deviator, which is not equal to the deviator of the stress rate.

invalid with the use of the mixed components. We therefore define the unsymmetric tensors $*\sigma'$ and $**\sigma'$ by means of

$$*\sigma'^{\beta}_{\alpha} = D_t \sigma_{\alpha}{}^{\beta} \tag{9}$$

$$**\sigma'^{\beta}{}_{\alpha} = D_t \sigma^{\beta}{}_{\alpha} \tag{10}$$

and the symmetric tensor[6]

$$\sigma' = \tfrac{1}{2}(*\sigma' + **\sigma'). \tag{11}$$

The Cartesian components of this tensor are given by

$$\sigma'_{ij} = D_t \sigma_{ij} - \omega_{ki}\sigma_{jk} - \omega_{kj}\sigma_{ik} \tag{12}$$

in which

$$\omega_{ij} = \tfrac{1}{2}(v_{i,j} - v_{j,i}) \tag{13}$$

are the components of the skew-symmetric vorticity tensor.

It is seen that σ' is Jaumann's definition; in other words, the artifice of introducing a local rigid coordinate system moving with the particle as employed by Jaumann and again recently by Oldroyd [6] is equivalent to the use of the symmetrized convected time derivative of the mixed components. Note also that σ' is the mean of σ'' and σ'''. An analogous definition applied to vectors leads to a rate definition which is in agreement with elementary concepts.

Under certain circumstances, the use of a weighted tensor in the constitutive equations may be more convenient. For example, Drucker's argument [9] establishing the convexity of the plastic loading function may lead to the conclusion that this function should be expressed in terms of the Kirchhoff tensor components, since the latter govern the rate of work per fixed unit of mass (rather than that per time dependent unit of volume). The equivalent Cartesian stress rate, referred to the instantaneous system, is given by

$$\sigma^{IV}{}_{ij} = D_t \sigma_{ij} - \omega_{ki}\sigma_{jk} - \omega_{kj}\sigma_{ik} + v_{k,k}\sigma_{ij} \tag{14}$$

which is acceptable in the light of the preceding discussion.

It may be of some interest to discuss briefly the question of the definition of the strain rate. Offhand, it appears as if similar arguments should make it necessary to define the latter also in terms of the mixed components. This is actually not the case since the definition of the strain tensor $\eta_{\alpha\beta}$ itself employs the metric tensor and, consequently, some of the resulting relationships become somewhat simpler. In fact, with the strain deviator defined by

$$h_{\alpha\beta} = \eta_{\alpha\beta} - \tfrac{1}{3}\eta_{\gamma\lambda} g^{\gamma\lambda} g_{\alpha\beta} \tag{15}$$

then its rate, in view of Eq. (8), takes the form

$$e_{\alpha\beta} = D_t h_{\alpha\beta} = (\epsilon_{\alpha\beta} - \tfrac{1}{3}\epsilon^{\gamma}_{\gamma} g_{\alpha\beta}) - \tfrac{2}{3}(\eta^{\gamma}_{\gamma}\epsilon_{\alpha\beta} - \eta^{\gamma\lambda}\epsilon_{\gamma\lambda} g_{\alpha\beta}). \tag{16}$$

[6]Of course the symmetry of σ' is a consequence of that of σ. Similarly Eq. (11) defines an antisymmetric tensor if the original tensor is also antisymmetric. In particular, if the operation is performed on ω itself, then (as is plausible from elementary considerations) the material time derivative coincides with the convected derivative. This fact is useful in the determination of higher order rates.

In this case the vanishing of the strain rate components $\epsilon_{\alpha\beta}$ implies also that the deviator components $h_{\alpha\beta}$ are stationary. Moreover, a limited inverse is also true; that is, the vanishing of $e_{\alpha\beta}$ implies that the strain rate is isotropic.

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STEADY SPHEROIDAL VORTICES—MORE EXACT SOLUTIONS TO THE NAVIER-STOKES EQUATION*

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Abstract. The vorticity equation, the curl of the Navier-Stokes equation, is considered in ellipsoidal coordinates. The steady spheroidal vortex solutions are demonstrated as examples of a class of exact flow solutions characterized by a simple linear vorticity distribution.

Introduction. The Navier-Stokes equation expresses the conservation of momentum in a form suitable for continuum fluid mechanics. For incompressible fluid flow the system of the continuity equation plus the Navier-Stokes equation has a solution that is completely determined, in principle. However, solutions that satisfy the equations exactly are rare due to the non-linear convective term. One such solution that should be included in the meager list of exact solutions is the Hill-Hadamard spherical vortex.

The spherical vortex is an example of a class of steady axi-symmetric flow solutions consisting of a rotational part where the vorticity is proportional to the distance from the axis and an irrotational part. Other examples of this class will be given, in particular, the steady oblate spheroidal vortices which appear to be physically significant.

Hill-Hadamard spherical vortex. Hill [1] showed in 1894 that the stream function

$$\psi = \sin^2 \theta(Ar^2 + Br^4) \tag{1}$$

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satisfied the inviscid condition for steady motion (Lamb [2], p. 245)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = \mathsf{D}\psi = \varpi^2 f(\psi), \tag{2}$$

where (x, ϖ, ϕ) is the cylindrical polar coordinate system. The relation to the spherical polar coordinate system (r, θ, ϕ) is given by $x = r\cos\theta, \varpi = r\sin\theta$. Hill's solution takes $f(\psi)$ as a constant $= -10B$.

Hadamard [3] (and Rybczynsky [4]) in 1911, while discussing the viscous motion of a fluid sphere within another fluid, used the same stream function (1). Stokes [5] had given

$$\psi = \sin^2\theta(Ar^2 + Br^4 + Cr + Dr^{-1}) \tag{3}$$

as a solution to the steady viscous vorticity equation

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\right\}^2\psi = \mathsf{D}^2\psi = 0. \tag{4}$$

The physical requirement of finite velocities as $r \to 0$ eliminated the two lower powers of r.

Thus it has been known for a long time that this stream function (1) independently satisfies a condition derived from the Navier-Stokes equation under the assumption of purely inviscid flow (2) and the condition derived for purely viscous flow (4). It follows, though never previously shown to the author's knowledge, that the same stream function satisfies the complete Navier-Stokes equation. Therefore, it is an exact solution.

The velocity field follows from the derivatives of ψ. $\mathbf{q} = u\mathbf{i}_x + v\mathbf{i}_\varpi$ where

$$u = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}, \qquad v = \frac{1}{\varpi}\frac{\partial\psi}{\partial x}.$$

The pressure field is found by integration of the momentum equation written in the "vorticity form"

$$\frac{\partial\mathbf{q}}{\partial t} - \mathbf{q}\times\boldsymbol{\omega} = -\nabla\left(\frac{p}{\rho} + \Omega + \frac{1}{2}\mathbf{q}\cdot\mathbf{q}\right) - \nu\nabla\times\boldsymbol{\omega} \tag{5}$$

with the vorticity vector $\boldsymbol{\omega} = \nabla\times\mathbf{q} = \omega\mathbf{i}_\phi = (1/\varpi\mathsf{D}\psi)\mathbf{i}_\phi$. Whence

$$\begin{aligned}\frac{p}{\rho} + \Omega + \frac{1}{2}q^2 &= 10B\{A\varpi^2 + B\varpi^2(x^2 + \varpi^2) - 2\nu x\} \\ &= 10B\{(Ar^2 + Br^4)\sin^2\theta - 2\nu r\cos\theta\}.\end{aligned}$$

For a spherical vortex $\psi = 0$ at $r = a$. This requires $A = -Ba^2$. The Hill-Hadamard spherical vortex is one of the class of flow solutions consisting of a rotational part where the vorticity ω is proportional to ϖ and an irrotational part. Spherical vortices have been demonstrated in real fluids by Spells [6] and Savic [7].

Poiseuille flow. A more familiar example of this class of exact flow solutions with a linear vorticity distribution is the well-known Poiseuille flow. This is often quoted as an exact solution, but the vorticity distribution receives little attention. Its stream function can be written (a = pipe radius)

$$\psi = A\varpi^2(a^2 - \varpi^2). \tag{6}$$

This can be rewritten into a rotational part and an irrotational part as

$$\psi = \left\{-\frac{6A}{5}r^4\sin^2\theta\right\} + \left\{Aa^2r^2C_2^{-1/2}(\cos\theta) + \frac{A}{5}r^4C_4^{-1/2}(\cos\theta)\right\}.$$

(vorticity $\omega = 12\,Ar\sin\theta$)

The irrotational part is given in terms of the Gegenbauer polynomials $C_n^{-1/2}(\cos\theta)$ after Savic [7] rather than the more usual Legendre polynomial form for reasons that will become apparent in the next section. The validity of this solution for real fluids in laminar flow is well-established.

Exact solution in oblate ellipsoidal coordinates. It will be demonstrated that the steady spheroidal vortex is another example of the same class of exact flow solutions. For this purpose the pertinent equations are first transformed to the oblate ellipsoidal coordinate system (η, β, ϕ) by means of the relations

$$\begin{aligned} x &= \sinh\eta\cos\beta \\ y &= \cosh\eta\sin\beta\cos\phi \\ z &= \cosh\eta\sin\beta\sin\phi \qquad (\varpi = \cosh\eta\sin\beta) \end{aligned} \tag{7}$$

The continuity equation becomes (see Goldstein [8], p. 114)

$$\frac{\partial}{\partial\eta}[\cosh\eta\sin\beta(\sinh^2\eta+\cos^2\beta)^{1/2}u_\eta]+\frac{\partial}{\partial\beta}[\cosh\eta\sin\beta(\sinh^2\eta+\cos^2\beta)^{1/2}v_\beta]=0 \tag{8}$$

for $\mathbf{q} = u_\eta\mathbf{i}_\eta + v_\beta\mathbf{i}_\beta$.

Define a current stream function ψ by

$$\begin{aligned} u_\eta &= -\frac{(\cosh\eta\sin\beta)^{-1}}{(\sinh^2\eta+\cos^2\beta)^{1/2}}\frac{\partial\psi}{\partial\beta}, \\ v_\beta &= \frac{(\cosh\eta\sin\beta)^{-1}}{(\sinh^2\eta+\cos^2\beta)^{1/2}}\frac{\partial\psi}{\partial\eta}, \end{aligned} \tag{9}$$

to identically satisfy (8). The steady vorticity equation can then be written (see [8], p. 115)

$$-\frac{(\cosh\eta\sin\beta)^{-1}}{(\sinh^2\eta+\cos^2\beta)}\left\{\frac{\partial(\psi,\mathsf{D}\psi)}{\partial(\eta,\beta)}-\frac{2\,\mathsf{D}\psi}{\cosh\eta\sin\beta}\frac{\partial(\psi,\cosh\eta\sin\beta)}{\partial(\eta,\beta)}\right\}=\nu\,\mathsf{D}^2\psi, \tag{10}$$

where

$$\mathsf{D}=(\sinh^2\eta+\cos^2\beta)^{-1}\left(\frac{\partial^2}{\partial\eta^2}-\tanh\eta\frac{\partial}{\partial\eta}+\frac{\partial^2}{\partial\beta^2}-\cot\beta\frac{\partial}{\partial\beta}\right).$$

Let

$$\psi = \psi_1\,(\text{irrotational}) + \psi_2\,(\text{rotational}). \tag{11}$$

The irrotational part ψ_1, where

$$\omega=\frac{1}{\varpi}\mathsf{D}\psi_1=0$$

represents a trivial solution to (10) but it is convenient to discuss this first.

Assume $\psi_1 = f(\eta)g(\beta)$ and $\mathsf{D}\psi_1 = 0$ is separable for $n(n-1) > 0$ into the pair of ordinary differential equations

$$f''(\eta)-\tanh\eta f'(\eta)-n(n-1)f(\eta)=0; \qquad g''(\beta)-\cot\beta g'(\beta)+n(n-1)g(\beta)=0.$$

By applying the transformation $i\sinh\eta = z$ to the former and $\cos\beta = z$ to the latter, *both* equations reduce to a Gegenbauer differential equation (see Magnus and Oberhettinger [9], p. 76)

$$(1-z^2)P''(z)+n(n-1)P(z)=0. \tag{12}$$

A solution for this is $P = C_n^{-1/2}(z)$, the Gegenbauer polynomial of order n and degree $(-\frac{1}{2})$.

Thus a general solution ψ_1 valid for regions containing the origin is

$$\psi_1 = \sum_{n=2}^{\infty} A_n C_n^{-1/2}(\cos\beta) C_n^{-1/2}(i \sinh\eta). \tag{13}$$

The $n = 0, 1$ terms have been dropped because they lead to velocity singularities along the axis. The Gegenbauer functions of the second kind which also satisfy Eq. (12) lead to infinite velocities as $\beta \to \pi/2$ or $\eta \to 0$ and thus do not appear as part of the physical flow solution. However, the Gegenbauer functions of the second kind with arguments a function of η would be important for other flow problems than the type considered here.

Consider now a rotational part

$$\psi_2 = B \sin^2\beta \cosh^2\eta (\cos^2\beta - \sinh^2\eta). \tag{14}$$

This makes both the inviscid part of the vorticity equation and the viscous part identically zero. $\omega\varpi = \mathsf{D}\psi_2$ can be transformed by $\cos\beta = \mu$, $i \sinh\eta = z$ into

$$(\mu^2 - z^2)^{-1}\left[(1-\mu^2)\frac{\partial^2}{\partial\mu^2} - (1-z^2)\frac{\partial^2}{\partial z^2}\right]B(1-\mu^2)(1-z^2)(\mu^2+z^2).$$

Carrying out the indicated operation, this equals

$$\begin{aligned} -10B(1-\mu^2)(1-z^2) &= -10B\sin^2\beta\cosh^2\eta \\ &= \kappa\varpi^2. \end{aligned}$$

(Or $\omega = \kappa\varpi$.)

The inviscid part of the vorticity equation (10) gives

$$-\frac{(\cosh\eta\sin\beta)^{-1}}{(\sinh^2\eta + \cos^2\beta)}\left\{\kappa\begin{vmatrix} \frac{\partial\psi_2}{\partial\eta} & \frac{\partial\psi_2}{\partial\beta} \\ \frac{\partial\varpi^2}{\partial\eta} & \frac{\partial\varpi^2}{\partial\beta}\end{vmatrix} - 2\kappa\varpi\begin{vmatrix} \frac{\partial\psi_2}{\partial\eta} & \frac{\partial\psi_2}{\partial\beta} \\ \frac{\partial\varpi}{\partial\eta} & \frac{\partial\varpi}{\partial\beta}\end{vmatrix}\right\}$$

which is identically zero. The viscous part $\nu\mathsf{D}^2\psi_2$ is also satisfied because the term equivalent to $\mathsf{D}\psi_2$ appears in the general solution ψ_1 for $n = 2$ or $\mathsf{D}^2\psi_2 = 0$.

The oblate spheroidal vortex. The exact solution $\psi = \psi_1 + \psi_2$ with the boundary conditions $\psi = 0$ on the ellipsoid $\eta = \eta_0$ and that there be stagnation points on the central axis yields:

$$\psi = C\sin^2\beta\cosh^2\eta\left\{\sinh^2\eta_0 - \sinh^2\eta + \frac{\cos^2\beta}{\sinh^2\eta_0}(\sinh^2\eta_0 - \sinh^2\eta)\right\}. \tag{15}$$

This is a spheroidal vortex. The velocity follows directly from the derivatives of ψ and the pressure field can be obtained by integration of the momentum equation in the vorticity form. The component momentum equations in this coordinate system are

$$\begin{aligned} -v_\beta\omega = &-(\sinh^2\eta + \cos^2\beta)^{-1/2}\frac{\partial}{\partial\eta}\left(\frac{p}{\rho} + \Omega + \tfrac{1}{2}[u_\eta^2 + v_\beta^2]\right) \\ &+ \nu(\sinh^2\eta + \cos^2\beta)^{-1/2}\left(\omega\cot\beta + \frac{\partial\omega}{\partial\beta}\right), \\ u_\eta\omega = &-(\sinh^2\eta + \cos^2\beta)^{-1/2}\frac{\partial}{\partial\beta}\left(\frac{p}{\rho} + \Omega + \tfrac{1}{2}[u_\eta^2 + v_\beta^2]\right) \\ &-\nu(\sinh^2\eta + \cos^2\beta)^{-1/2}\left(\omega\tanh\eta + \frac{\partial\omega}{\partial\eta}\right). \end{aligned}$$

The pressure field is

$$\frac{p}{\rho} + \Omega + \tfrac{1}{2}(u_\eta^2 + v_\beta^2) = \kappa\{-\psi + 2\nu \cos\beta \sinh\eta\},$$

where $\kappa = -10C$ is the proportionality factor between the vorticity ω and the axial distance ϖ.

Transforming back to the cylindrical coordinate system

$$\psi = C\varpi^2(\cosh^2\eta_0 - \varpi^2 - x^2\coth^2\eta_0), \tag{16}$$

where $\psi = 0$ on the oblate ellipsoid

$$x^2(\sinh^2\eta_0)^{-1} + \varpi^2(\cosh^2\eta_0)^{-1} = 1.$$

An example is shown in Fig. 1 for an ellipsoid with fineness ratio $= \tanh\eta_0 = .5(\eta_0 = 0.55)$. The velocity components parallel and normal to the axis are

$$u = -2C(\cosh^2\eta_0 - 2\varpi^2 - x^2\coth^2\eta_0),$$

$$v = -2C(x\varpi\coth^2\eta_0).$$

In the axial plane, $x = 0$, note a circle of zero velocity. This center of circulation occurs at the same fixed ratio of the semi-major radius, $(2)^{-1/2}$, for all values of η_0.

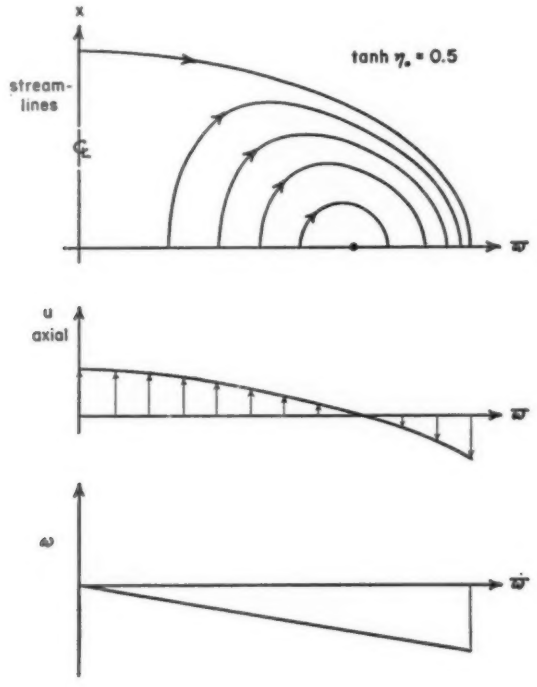

Fig. 1
An example of a spheroidal vortex

Real vortices often have boundaries that can be approximated by oblate spheroids. A comparison of some real flows with theory is given in another place, O'Brien [10]. The measured centers of circulation of quasi-steady vortices over a fairly wide Reynolds number range compare reasonably well with the theoretical value given here.

A more general viscous solution. Neglecting the inviscid part of the steady vorticity equation, a rather general solution, ψ_2, can be determined from the relation $D\psi_2 = \psi_1$. Although the equation is not separable, the rotational solution in oblate ellipsoidal coordinates (valid for the region containing $\eta = 0$) has been found to be

$$\psi_2 = \sum_{n=2}^{\infty} B_n C_n^{-1/2}(\cos\beta) C_n^{-1/2}(i \sinh\eta)(\cos^2\beta - \sinh^2\eta). \tag{17}$$

But for $n \geq 3$ this stream function does not simultaneously satisfy the inviscid part.

The prolate spheroidal vortex. In similar fashion the problem can be reformulated in prolate ellipsoidal coordinates. The stream function for a prolate spheroidal vortex is

$$\psi = C\varpi^2(\sinh^2\xi_0 - \varpi^2 - x^2\tanh^2\xi_0). \tag{18}$$

In analogy to the oblate cases, the center of circulation is at the same fixed ratio, $(2)^{-1/2}$, of the midsection radius for all ξ_0.

Conclusion. It is not known whether there are other members of this class of flow solutions. The second order Laplacian equation that governs "perfect" fluid flow is separable in just eleven coordinate systems. The fourth order purely viscous vorticity equation (4) is not even separable in the two coordinate systems considered here (oblate ellipsoidal and prolate ellipsoidal). Assume, by use of another coordinate system, a new solution to this linear equation is found. If it has vorticity proportional to the axial distance, ϖ, it will be another exact solution.

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A NOTE ON UNIQUENESS IN CLASSICAL ELASTODYNAMICS*

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Introduction. The conventional uniqueness theorems of classical elasticity theory, first established by Kirchhoff [1][1] for the equilibrium case and later extended to elastodynamics by Neumann [2], rest on the assumption that the elastic constants satisfy inequalities which are necessary and sufficient for the positive definiteness of the strain-energy density. The theorems referred to thus presuppose that

$$\mu > 0, \qquad -1 < \sigma < \tfrac{1}{2}, \tag{1}$$

if μ and σ respectively denote the shear modulus and Poisson's ratio of the material.

Actual elastic materials obeying the linear isotropic stress-strain law conform to (1) and, in fact, fail to exhibit negative Poisson ratios. Nevertheless, the question as to whether conditions (1) are *necessary* for uniqueness, is of obvious theoretical interest. It is the purpose of the present note to show, with limitation to the first boundary-value problem of elastodynamics (surface displacements prescribed) and for bounded domains, that the inequalities (1) may be relaxed without loss of uniqueness. Specifically, we find that in this instance (1) may be replaced by the less stringent assumption

$$\mu \geq 0, \qquad -\infty < \sigma < \tfrac{1}{2}, \qquad 1 \leq \sigma < \infty, \tag{2}$$

or, equivalently, by

$$c_1^2 = \frac{\mu}{\rho} \geq 0, \qquad c_2^2 = \frac{2(1-\sigma)\mu}{(1-2\sigma)\rho} \geq 0, \tag{3}$$

in which c_1 and c_2 are the velocities of equivoluminal and irrotational waves in a medium of unlimited extent, whereas ρ stands for the positive mass density.[2] Accordingly, the generalized inequalities (2) admit a simple intrinsic interpretation: they require that the two basic wave-velocities of the elastic solid be real.

The foregoing extension of Neumann's theorem, which will be proved subsequently, is the dynamic counterpart of a known extension of the uniqueness theorem in elastostatics. The solution to the first boundary-value problem in the equilibrium theory is unique (under suitable regularity assumptions) if, instead of (1), one stipulates merely that

$$\mu \neq 0, \qquad -\infty < \sigma < \tfrac{1}{2}, \qquad 1 < \sigma < \infty. \tag{4}$$

The relaxed inequalities (4) are evidently met provided the two wave velocities c_1 and c_2 are real and non-zero. For a bounded domain, this generalization of Kirchhoff's uniqueness theorem is implicit already in observations due to Kelvin [3]; it may also be inferred from available results appropriate to the Dirichlet problem for strongly

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[1]Numbers in brackets refer to the list of publications at the end of this paper.

[2]Actually, the inequalities (3) are sufficient for uniqueness in the present circumstances even if ρ is negative.

elliptic systems of partial differential equations,[3] since (4) hold if and only if the displacement equations of equilibrium are of the strongly elliptic type. As shown by Duffin and Noll [6],[4] the inequalities (4) assure the uniqueness of the solution to the first problem of elastostatics also for unbounded exterior domains, provided the displacements are required to vanish uniformly at infinity. Finally, it is clear from counter-examples due to Ericksen and Toupin [8] and Ericksen [9] that (4), in the equilibrium case, cannot be weakened further without loss of uniqueness.

A generalization of the uniqueness theorem in elastodynamics. The displacement equations of motion in the linear theory of homogeneous and isotropic elastic solids take the form

$$\mu\nabla^2\mathbf{u} + \frac{\mu}{1-2\sigma}\nabla\nabla\cdot\mathbf{u} + \mathbf{f} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{5}$$

in which $\mathbf{u}(x, t)$ is the displacement vector[5] and $\mathbf{f}(x, t)$ is the body-force intensity vector, the single argument x representing the triplet of cartesian coordinates and t denoting the time. Equation (5), which must hold throughout the region of space D occupied by the medium, is subject to the initial conditions

$$\mathbf{u}(x, 0) = \mathbf{u}^0(x), \left.\frac{\partial\mathbf{u}}{\partial t}\right|_{(x,0)} = \mathbf{v}^0(x) \quad \text{in} \quad D, \tag{6}$$

where $\mathbf{u}^0(x)$ and $\mathbf{v}^0(x)$ are the prescribed initial distributions of displacement and velocity. In the first problem of elastodynamics the accompanying boundary conditions are characterized by

$$\mathbf{u}(x, t) = \mathbf{u}^*(x, t) \quad \text{on} \quad B \qquad (0 < t < \infty), \tag{7}$$

if B is the boundary of D, while $\mathbf{u}^*(x, t)$ are the given surface displacements.

We now prove the following theorem. *Let D be a bounded regular region of space*[6] *with the boundary B and suppose that* (3) *hold. Then there exists at most one* $\mathbf{u}(x, t)$, *twice continuously differentiable for* (x_1, x_2, x_3) *in* $D + B$ $(0 \leq t < \infty)$, *which satisfies* (5) *in* $D(0 < t < \infty)$, *and meets* (6), (7).

In view of the linearity of (5), (6), (7) it is evidently sufficient to show that

$$\mu\nabla^2\mathbf{u} + \frac{\mu}{1-2\sigma}\nabla\nabla\cdot\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \quad \text{in} \quad D \qquad (0 < t < \infty), \tag{8}$$

together with

$$\mathbf{u}(x, 0) = \left.\frac{\partial\mathbf{u}}{\partial t}\right|_{(x,0)} = 0 \quad \text{in} \quad D, \tag{9}$$

$$\mathbf{u}(x, t) = 0 \quad \text{on} \quad B \qquad (0 < t < \infty), \tag{10}$$

imply $\mathbf{u}(x, t) = 0$ in $D(0 < t < \infty)$.

Let $K(t)$ be the kinetic energy of the entire body, so that

$$K(t) = \frac{\rho}{2}\int_D \left[\frac{\partial\mathbf{u}}{\partial t}\right]^2 dV, \tag{11}$$

---

[3] See Browder [4] and Morrey [5].

[4] See also [7].

[5] Letters in boldface designate vectors.

[6] The term "regular region of space" is used in the sense of Kellogg [10].

and define $G(t)$ through

$$G(t) = \frac{\mu}{2}\int_D [\kappa(\nabla\cdot\mathbf{u})^2 + (\nabla\times\mathbf{u})^2]\,dV, \qquad \kappa = \frac{2(1-\sigma)}{1-2\sigma}. \tag{12}$$

From the divergence theorem, with the aid of (8), (11), (12), and elementary vector-analytic identities, follows

$$\mu\int_B \left[\kappa\frac{\partial\mathbf{u}}{\partial t}\nabla\cdot\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t}\times(\nabla\times\mathbf{u})\right]\cdot\mathbf{n}\,dA = \frac{dK}{dt} + \frac{dG}{dt}, \tag{13}$$

where $\mathbf{n}$ is the outward unit normal of B.

According to (13) and (10),

$$K(t) + G(t) = c \qquad (0 < t < \infty), \tag{14}$$

in which c is a constant. But by (11), (12), (9),

$$K(0) = G(0) = 0, \tag{15}$$

whence, from (14) and continuity,

$$K(t) + G(t) = 0 \qquad (0 \le t < \infty). \tag{16}$$

Since $\kappa \ge 0$ by virtue of (3), both $K(t)$ and $G(t)$ are either non-negative or non-positive, so that (16) yields

$$K(t) = 0, \qquad G(t) = 0 \qquad (0 \le t < \infty). \tag{17}$$

On the other hand, the first of (17) in conjunction with (11) leads to

$$\frac{\partial\mathbf{u}}{\partial t} = 0 \quad \text{in} \quad D \qquad (0 \le t < \infty), \tag{18}$$

from which, because of (9), $\mathbf{u}(x, t) = \mathbf{u}(x, 0) = 0$ in $D(0 \le t < \infty)$. This completes the proof.

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BOOK REVIEWS

(*Continued from p. 142*)

to be informed about the present state of the art, and the practicing engineer who has to solve a given concrete problem. The latter will find a wealth of useful material in the form of tables, graphs, formulas and references. To the first the limitation of the general chapters to linearized theory will probably be somewhat disappointing. For both groups, however, the book will prove to be an indispensable tool.

H. PARKUS

Stochastic population models in ecology and epidemiology. By M. S. Bartlett. John Wiley & Sons, Inc., New York, and Methuen & Co. Ltd., London, 1960. x + 90 pp. $2.00.

This monograph discusses models for biological populations in ecological and epidemiological contexts where a full stochastic formulation is advisable. It is noteworthy that even such simplified models as are presented imply complex consequences. First various models to determine the frequency distributions for single species are presented; the Poisson model is the simplest of these; birth-and-death processes and the problem of extinction are next discussed. Growth and interaction, and competition, are complications leading to more complicated models. In the concluding chapters, epidemiological models are considered for the vicissitudes of animal or human populations subject to infection by some disease transmitted from individual to individual. The restriction to a closed population is relaxed to permit recurrent epidemics and in the last chapter, epidemiological problems are noticed in which the spatial or topographical factor is relevant.

The work will appeal to biometricians with the necessary background, to theoretically minded ecologists and other interested research workers in biology and medicine.

W. FREIBERGER

Automatic language translation. By Anthony G. Oettinger, with a Foreword by Joshua Whatmough. Harvard University Press, Cambridge, Mass., 1960. xix + 380 pp. $10.00.

About a dozen projects in mechanical language translation are being currently conducted in the United States alone; the great majority of them have, as their ultimate aim, a reasonably accurate and smooth translation from Russian into English, mostly in scientific and technical fields. This is also the case with the research being conducted at the Computation Laboratory at Harvard University of which Mr. Oettinger is the principal investigator and this book a thorough and informative progress report.

The book under review here deals with only one of the major problems which have to be resolved in machine translation research: the compilation of an adequate bilingual dictionary in a form which would make it readily accessible to the computer (stored on magnetic tape in the Harvard project), and the devising of programming procedures through which the words of a running text could be accurately matched with entries in this dictionary, thus achieving one of the essential steps of translation, the finding of lexically equivalent or near-equivalent elements in two languages. Once this step has been mastered, a rough approximation of the original text in another language, a so-called word-by-word translation, can be achieved. Word-by-word translations are of some use but are basically unsatisfactory because they contain no information about the grammatical properties of words or their syntactic relationship (except what may be communicated through lexical characteristics or the preservation of the original order of items), not even speaking of their complete disregard of such major problems as multiple meanings of words, idiomatic and other non-equivalent constructions, correct word-order, etc. Nevertheless, the compilation and operation of an automatic bilingual dictionary is a prerequisite to research in the other essential aspects of translation by data-processing machines, be it the mechanical analysis of the grammar and syntax of the source langugage, or the generating of grammatically and syntactically acceptable strings in the language into which the translating is being done. Oettinger discusses these other aspects only in general terms, pointing out that the Harvard project has not

progressed beyond the initial stages in such research. However, the fact that this book deals primarily with the compilation and operation of an automatic dictionary, possibly one of the more easily manageable of the major steps in mechanical translation, does not detract from its importance. In spite of the occasional optimistic newspaper reports, the time when a good translation can be obtained entirely by automatic means is not exactly around the corner. Thus, the thorough treatment which Oettinger gives to the work already accomplished at Harvard and his sober discussion of the limitations and difficulties which still must be overcome in the future will be appreciated by those researchers interested in automatic linguistic analysis and translation, who are aware of all the complexities of this field of research.

The book is divided into ten chapters, which are interspersed with numerous tables, diagrams, appendices, and reproductions of actual machine-produced lists. Some chapters also contain special bibliographical information in addition to the sizeable bibliography at the end of the volume. Although the author tried to make the discussion of information-processing machines and their programming as general as possible, the book is unavoidably strongly oriented toward the Univac I computer which the Harvard project has been using, and some of the special characteristics as well as the limitations of this machine have determined the approach to individual problems.

The first chapter is a general introduction to the organization and programming of data-processing machines, with the Univac system serving to illustrate the individual points. The chapter begins with a very elementary outline of the principles of such machines, obviously intended for the complete beginner in the field, but the speed with which the subsequent complex information is introduced is such that one may doubt whether a reader who has never studied a computer manual would be able to follow the explanation for more than a few pages. The first chapter also contains several sample problems and a number of exercises (without solutions supplied). The exercises, in the later chapters, get progressively rarer until, toward the end of the book, their occasional appearance in the midst of the discussion cannot but impress the reader as a distinct afterthought. I found this apparent hesitation about the format of the book, and the sporadic and not very effective attempts to give it a bit of a textbook character, rather distracting.

The second chapter, which deals with the structure of signs and pays particular attention to the properties of sets, is well organized and informative but still comprehensible to the non-specialist. It also contains a description of the Cyrillic Unityper, a specially adapted typewriter used for transcribing characters onto magnetic tape. The machine serves, in lower-case, for encoding Russian characters while, in upper-case, the usual layout of Roman letters is maintained. Chapter three discusses flow charts and automatic programming, and includes an illustrative flow chart and programming information for the transformation of the computer representation of Russian characters into one of the commonly used transliteration systems of Russian through letters of the English alphabet. Chapter four investigates the various aspects of a translation process and states the assumptions which make mechanical translation a realistic, even if complex, possibility. The bulk of the book, chapters five to nine, are devoted to the compilation and operation of a dictionary. The Harvard Automatic Dictionary contained, at the time this book was written, some 22,000 stem entries (it has apparently been enlarged to 30,000 since then). One stem entry does not necessarily correspond to a word in the conventional sense; one word may be represented by one or more stem entries, depending on the degree and complexity of the inflection which it undergoes. However, the Harvard Automatic Dictionary is not a "paradigm dictionary" which would list all the inflected forms of a word separately, but rather something of a "canonical dictionary", containing only those stems which are necessary for the successful matching of the strings in actual text and, of course, a list of affixes, so that strings of the text can be factored before they are matched with the dictionary entries. The dictionary is compiled in several steps, some of which are automatic. The appropriate words are selected for inclusion from conventional dictionaries and other sources, and then manually classified into their various word-classes (nouns, verbs, etc.) and inflectional sub-classes. This classificatory information is included when the words are transcribed onto magnetic tape. The computer, supplied with a list of inflectional endings and a program for applying them, generates the complete paradigm of each word (if that word can indeed be inflected); after this has been accomplished, all the forms now in the list are automatically factored into stem and affix, duplications eliminated, and the final list of distinct stems entered in the dictionary, together with the English equivalents which have been assigned to them by a human translator. The running text which is to be analyzed is processed in a similar fashion, with words factored into stems and affixes. Since the same algorithm is used in the generation of stems which become entry keys in the dictionary and in

the operation to factor strings of the analyzed text, there is a guarantee that a matching stem will be found for each word of the text, provided, of course, that such a word had been originally processed for inclusion in the dictionary.

The final format of an entry in the automatic dictionary consists of 30 machine words (of 12 characters per word) which include, besides the Russian stem and one or more of its English equivalents, the class designation of the Russian item and various other information, such as the specification of verbal aspect or syntactic government by verbs, which are intended for use in the subsequent automatic grammatical analysis.

Mr. Oettinger also gives an actual example of a word-by-word translation (which he calls a "trot"), containing all the possible English meanings included in the dictionary for every Russian word. Through manual post-editing which supplies some grammatical information and selects the appropriate English alternative, the "trot" can result in a useful, even if inelegant, translation.

The last chapter, with the descriptive title "From dictionary to translation", discusses the future plans of the Harvard project which will hopefully lead to an acceptable mechanical translation of Russian texts.

The book is well organized, with some minor exceptions. The numerous tables and diagrams, which are scattered throughout, are referred to in the text by their numbers rather than by pages on which they occur; this results in a tedious search when they are mentioned in various places in the book. The index which completes the volume leaves something to be desired.

As a whole, this book is a competent and informative contribution to a complex field of research. As mentioned previously, it deals, in a detailed manner, only with one of the essential aspects of automatic translation, the lexical one, and it is careful to avoid any implications that the other major steps which still must be tackled are about to be resolved. After reading the book, one is left with the impression that research in automatic translation has made considerable strides in the last few years but that, all the same, the day when one could have a secretary copy a Russian article onto magnetic tape and get, in a few minutes, a good English translation from the computer is not yet within sight.

The foreword by Joshua Whatmough, which is promisingly announced on the title page and on the dust-jacket, is only a brief statement of somewhat whimsically expressed praise for the book.

HENRY KUCERA

A guide to mathematical tables. By A. V. Lebedev and R. M. Fedorova. English edition prepared from the Russian by D. G. Fry. Pergamon Press, New York, 1960. xlvi + 586 pp. $15.00.

The tables listed are arranged under the following headings: Powers, rational and algebraic functions—Trigonometric functions. Various values connected with the wide and the sphere—Exponential and hyperbolic functions—Common and natural logarithms—Factorials, Euler integrals and related functions—Sine and cosine integrals, exponential and logarithmic integrals and related functions—Probability integrals, probability distribution functions, and related functions—Elliptic integrals and functions—Legendre functions and polynomials—Cylinder functions—Certain special functions and integrals—Solutions of certain equations—Sums and values connected with finite differences—Mathematical constants—Primes, factors and products, quotients and fractions. While the text has been translated all tabular matter has been directly reproduced from the Russian original. This involves minor deviations from the customary notations (e. g. the use of tg for tan) but has the great advantage of excluding typographical errors that might have arisen if this material had been reset.

Stationary processes and prediction theory. By Harry Furstenberg. Princeton University Press, New Jersey, 1960. 283 pp. $5.00.

The question of the "predictability" of a sequence is examined in this study. Specifically, given a sequence of observations $\cdots, \xi(-n), \cdots, \xi(-1), \xi(o)$ (possibly nonnumerical), the notion of prediction of future possible values is considered. The discussion is highly motivated by the statistical theory of prediction (see the work of A. N. Kolmogorov, N. Wiener and their followers) but is formulated in rather abstract language. The sequence of observations can be thought of as the realization of a random

process in the past and present. If the realization is consistent with stationarity of a potential random mechanism generating it, a prediction in terms of a probability distribution on possible futures is proposed. The presentation rests heavily on the theory of commutative C*-algebras.

M. ROSENBLATT

The potential theory of unsteady supersonic flow. By J. W. Miles. Cambridge University Press, New York, 1959. xii + 220 pp. $8.50.

This monograph is a purely theoretical one, essentially on the unsteady aerodynamics of slender bodies in supersonic flow. The treatment is limited to frictionless potential flows and, almost entirely, to linearized flows. Within these bounds of subject matter the author has prepared a comprehensive and well-organized monograph. In the book are evident not only the results of the author's own extensive research in the field but also his thorough knowledge of the literature.

The first three chapters introduce the subject, by presenting the basic linearized equations and discussing their transformations. Chapter 4, treating reduction of the equations to those for steady flow, includes the reductions of Magnaradze and of Garner. Succeeding chapters treat in detail two-dimensional problems, simple planforms, the rectangular wing, the quadrilateral wing, slender wing methods, the delta wing, the low aspect ratio rectangular wing, and non-planar bodies. A final chapter on nonlinear problems and an appendix on reverse flow theorems close the monograph. A bibliography and three indices are provided.

This excellent monograph is one of a series of monographs on quite specialized topics in mechanics and applied mathematics. The availability in organized treatments of such material is of great value to those not already specialists in a field. Few indeed could easily get an understanding of the state of the art in the theory of unsteady supersonic flow from sources other than Miles' treatment. The editors are to be commended.

WALLACE D. HAYES

Digital computers and nuclear reactor calculations. By Ward C. Sangren. John Wiley & Sons, Inc., New York, London, 1960. xi + 208 pp. $8.50.

This book falls naturally into two parts: the first four chapters constitute an introduction to digital computers, programming, numerical analysis and reactor problems, and the last four discuss in some detail a particular reactor problem, namely fission-product poisoning, as a typical example of the application of computers to reactor physics.

The style is leisurely and most readable, the argument on the whole elementary. The book serves as an excellent survey for those without prior knowledge or specialised interest, or as an introduction with methods for generating random numbers obeying given probability distributions. In "Fourier to more advanced works, such as "*Numerical methods for nuclear reactor calculations*" by G. I. Marchuk (see Quart. Appl. Math., Vol. 18, p. 36), for anyone intending to work in the field.

WALTER FREIBERGER

Contributions to probability and statistics. Essays in honor of Harold Hotelling. Edited by I. Olkin, S. G. Ghurye, W. Hoeffding, W. G. Madow, H. B. Mann. Stanford University Press, California, 1960. x + 517 pp. $6.50.

This book is a collection of forty-two papers on probability and statistics, mostly mathematical statistics, dedicated to Harold Hotelling on his sixty-fifth birthday. Most of the papers are results of new research rather than expository papers.

G. F. NEWELL

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Contents

T. C. Benton: Three-dimensional flows inside a cylinder 81
P. M. Naghdi and W. L. Wainwright: On the time derivative of tensors in mechanics of continua 95
R. D. Mindlin and H. L. Cooper: On the equations of extensional motion of crystal plates 111
M. Lowengrub: Stress in the vicinity of a crack in a thick elastic plate . . 119
Z. Mróz: On a problem of minimum weight design 127
T. R. Goodman and R. Greif: Hydrodynamic inertia coefficients for a slender body with a sail 137

Notes:
P. G. Hodge, Jr.: A new interpretation of the plastic minimum principles 143
R. W. Newcomb: On the simultaneous diagonalization of two semi-definite matrices 144
D. Hazony: An upper bound on right half plane zeros 146
J. B. Thomas and E. Wong: The distribution of quadratic functionals 150
E. T. Kornhauser: Further extensions of Schuster's integral 153
E. E. Jones: Note on the Poincaré boundary-value problem 155
J. Jones, Jr.: On matrix differential equations 157
A. H. Zemanian: Further properties of certain classes of transfer functions: II 158
E. F. Masur: On the definition of stress rate 160
V. O'Brien: Steady spheroidal vortices—More exact solutions to the Navier-Stokes equation 163
M. E. Gurtin and E. Sternberg: A note on uniqueness in classical elastodynamics 169

Book Reviews 110, 136, 142, 172

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